

WALL CROSSING OF THE MODULI SPACES OF PERVERSE COHERENT SHEAVES ON A BLOW-UP.

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ABSTRACT. We give a remark on the wall crossing behavior of perverse coherent sheaves on a blow-up [21], [22] and stability condition [23]

0. INTRODUCTION

Let X be a smooth projective surface over an algebraically closed field k of characteristic 0. For $(\beta, \omega) \in \text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)_{\mathbb{R}}$, Arcara and Bertram [1] constructed stability conditions $\sigma_{(\beta, \omega)}$ such that the structure sheaves of points k_x ($x \in X$) are stable of phase 1. In [23], [24], Toda constructed new examples of stability conditions $\sigma_{(\beta, \omega)}$ which is regarded to an extension of Arcara and Bertram's examples to non-ample ω . Moreover Toda showed that new examples are related to stability conditions on blow-downs of (-1) -curves on X . Assume that ω is close to ample in $\text{NS}(X)_{\mathbb{R}}$. Then $\sigma_{(\beta, \omega)}$ is related to stability condition on a blow-down $\pi : X \rightarrow Y$ of a (-1) -curve C of X . In this case, we can set $\omega := \pi^*(L) + tC$, where $L \in \text{Amp}(Y)$ and $|t|$ is sufficiently small. If $t < 0$, then $\sigma_{(\beta, \omega)}$ is an example of Arcara and Bertram [1]. If $t = 0$, then instead of using a torsion pair of $\text{Coh}(X)$, $\sigma_{(\beta, \omega)}$ is constructed by using a similar torsion pair of a category of perverse coherent sheaves ${}^{-1}\text{Per}(X/Y)$ [5]. Since k_x ($x \in C$) becomes reducible in ${}^{-1}\text{Per}(X/Y)$, k_x is properly $\sigma_{(0, \omega)}$ -semi-stable. If $t > 0$, then k_x ($x \in C$) is not $\sigma_{(0, \omega)}$ -semi-stable. In this case, $\mathbf{L}\pi^*(k_y)$ is $\sigma_{(0, \omega)}$ -stable and $\sigma_{(0, \omega)}$ -semi-stable objects are parameterized by Y .

In this note, we shall give examples of Bridgeland semi-stable objects on X . In [21] and [22], Nakajima and the author studied the relation of moduli spaces of Gieseker semi-stable sheaves on Y and X by looking at the wall crossing behavior in the category of perverse coherent sheaves. As expected from Toda's papers and also the relation with the Gieseker semi-stability in the large volume limit, we shall show that Gieseker type semi-stability of perverse coherent sheaves corresponds to Bridgeland semi-stability in the large volume limit. Then we shall explain our wall crossing behavior of the moduli of perverse coherent sheaves in terms of Bridgeland stability condition (subsection 5.3).

For the proof of our results, we also need to study the large volume limit where ω is ample. As is proved by Bridgeland [6] and Lo and Qin [15], Bridgeland's stability $\sigma_{(\beta, \omega)}$ is related to (twisted) Gieseker stability if (ω^2) is sufficiently large (Proposition 6.11). On the other hand if an object $E \in \mathbf{D}(X)$ satisfies $(c_1(E) - \text{rk } E\beta_0, \omega_0) = 0$, then $\sigma_{(\beta_0, \omega_0)}$ -stability of E is related to μ -stability and the moduli space is related to the Uhlenbeck compactification of the moduli of μ -stable locally free sheaves. By looking at the wall and chamber structure near $\sigma_{(\beta_0, \omega_0)}$, we shall show that each adjacent chamber of (β_0, ω_0) contains a point at the large volume limit, which implies that each chamber corresponds to Gieseker semi-stability. Thus we may say that Bridgeland stability unifies (twisted) Gieseker stabilities and the μ -stability. We call these chambers *Gieseker chambers*. We explain the usual wall crossing of moduli spaces of Gieseker semi-stable sheaves [7], [9], [17] and also those of Uhlenbeck compactifications [11] in terms of Bridgeland stability conditions. The blow-up case is a slight generalization of this consideration.

Let us explain the organization of this article. In section 1, we introduce a bilinear form on the algebraic cohomology group $H^*(X, \mathbb{Q})_{\text{alg}}$, which is useful to state the Bogomolov inequality of μ -semi-stable sheaves. If X is a K3 surface or an abelian surface, this bilinear form is nothing but the Mukai's bilinear form. We next recall categories \mathfrak{C} of perverse coherent sheaves associated to a birational map $\pi : X \rightarrow Y$ of surfaces [28], and then introduce a stability condition by tilting categories of perverse coherent sheaves. In section 2, we consider the case of a blow-up $\pi : X \rightarrow Y$ of a smooth point of Y . We explain local projective generators of \mathfrak{C} and the moduli of semi-stable perverse coherent sheaves. We also explain some results in [21] and [22].

In section 3, we shall study wall and chamber structure of the moduli of σ -semi-stable objects $\mathcal{M}_{\sigma}(v)$, where v is the Chern character of the objects. We construct a map ξ from the subspace of stability conditions parameterized by $\text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)_{\mathbb{R}}$ to the positive cone of v^{\perp} with respect to the bilinear form on $H^*(X, \mathbb{Q})_{\text{alg}}$. Then we show that the wall and chamber structure on $\text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)_{\mathbb{R}}$ is the pull-back of a wall and chamber structure on v^{\perp} . In particular, all semi-stabilities on a fiber are the same. The wall and chamber structure in v^{\perp} is used to compare Gieseker chambers. In section 4, we shall partially generalize

results in the previous section to the blow-up case. We also construct an analytic neighborhood U of the origin of \mathbb{C}^2 which parameterizes stability conditions. It is used to study the wall crossing for the blow-up case.

In section 5, we first recall a homomorphism from v^\perp to $\mathrm{NS}(M_H(v))_{\mathbb{Q}}$, where $M_H(v)$ is the moduli spaces of Gieseker semi-stable sheaves. It is expected to be surjective modulo $\mathrm{NS}(\mathrm{Alb}(M_H(v)))_{\mathbb{Q}}$ under suitable conditions. Composing this homomorphism with the map constructed in section 3, we have a map $\mathrm{NS}(X)_{\mathbb{Q}} \times \mathrm{Amp}(X)_{\mathbb{Q}} \rightarrow \mathrm{NS}(M_H(v))_{\mathbb{Q}}$. By the work of Bayer and Macri [2], the image of Gieseker chamber is contained in the nef cone of the moduli space. Then we shall explain Matsuki-Wentworth wall crossing of Gieseker semi-stability in terms of Bridgeland stability conditions. Finally we explain the wall crossing of the moduli spaces of perverse coherent sheaves as a wall crossing of Bridgeland stability conditions. In Appendix, we give some technical results. In particular, we explain the moduli of semi-stable objects in the large volume limit, which is a generalization of [18]. We also give a trivial example of Gieseker chamber in section 6.3.

During preparation of this note, Bertram and Martinez informed us they also studied the wall crossing of twisted Gieseker stability by using Bridgeland stability, and get the same result of section 5.2 ([4]).

1. BACKGROUND MATERIALS

1.1. Basic notations. Let X be a smooth projective surface over an algebraically closed field k of characteristic 0. As in Mukai lattice on abelian surfaces, let us introduce a bilinear form on $H^*(X, \mathbb{Q})_{\mathrm{alg}} := \mathbb{Q} \oplus \mathrm{NS}(X)_{\mathbb{Q}} \oplus \mathbb{Q}$. For $x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in H^*(X, \mathbb{Q})_{\mathrm{alg}}$, we set

$$(1.1) \quad (x, y) := (x_1, y_1) - x_0 y_2 - x_2 y_0 \in \mathbb{Q}.$$

Let $\varrho_X := (0, 0, 1)$ be the fundamental class of X . We also use the notation $x = x_0 + x_1 + x_0 \varrho_X$ to denote $x = (x_0, x_1, x_2)$. For $x = (x_0, x_1, x_2)$, we set

$$(1.2) \quad \begin{aligned} \mathrm{rk} \, x &:= x_0, \\ c_1(x) &:= x_1. \end{aligned}$$

For $E \in \mathbf{D}(X)$, we set $v(E) := \mathrm{ch}(E) \in H^*(X, \mathbb{Q})_{\mathrm{alg}}$. Since

$$(1.3) \quad \begin{aligned} \Delta(E) &:= c_2(E) - \frac{\mathrm{rk} \, E - 1}{2 \mathrm{rk} \, E} (c_1(E)^2) \\ &= \frac{c_2(E^\vee \otimes E)}{2 \mathrm{rk} \, E} = \frac{(\mathrm{ch}(E)^2)}{2 \mathrm{rk} \, E}, \end{aligned}$$

the Bogomolov inequality is the following.

Lemma 1.1. *Assume that X is defined over a field of characteristic 0. Let E be a μ -semi-stable torsion free sheaf. Then $(v(E)^2) = 2(\mathrm{rk} \, E)\Delta(E) \geq 0$.*

Let L be an ample divisor on X .

$$P^+(X)_{\mathbb{R}} := \{x \in \mathrm{NS}(X)_{\mathbb{R}} \mid (x^2) > 0, (x, L) > 0\}$$

denotes the positive cone of X and $C^+(X)$ denotes $P^+(X)_{\mathbb{R}}/\mathbb{R}_{>0}$.

For a stability condition σ , $Z_\sigma : \mathbf{D}(X) \rightarrow \mathbb{C}$ is the central charge and \mathcal{A}_σ is the abelian category generated by σ -stable objects E with the phase $\phi_\sigma(E) \in (0, 1]$. Then σ consists of the pair $(Z_\sigma, \mathcal{A}_\sigma)$ of a central charge $Z_\sigma : \mathbf{D}(X) \rightarrow \mathbb{C}$ and an abelian category \mathcal{A}_σ . Let $\mathrm{Stab}(X)$ be the space of stability conditions. We have a map $\Pi : \mathrm{Stab}(X) \rightarrow H^*(X, \mathbb{C})_{\mathrm{alg}}$ such that $(\Pi(\sigma), *) = Z_\sigma(*)$. If σ satisfies the support property, then Π is locally isomorphic.

1.2. Perverse coherent sheaves. Let $\pi : X \rightarrow Y$ be a birational morphism of projective surfaces such that X is smooth, Y is normal and $R^1\pi_*(\mathcal{O}_X) = 0$. The notion of perverse coherent sheaves was introduced by Bridgeland [5]. Let us briefly recall a slightly different formulation of perverse coherent sheaves in [28]. Let G be a locally free sheaf on X which is a local projective generator of a category of perverse coherent sheaves \mathfrak{C} ([28]). Thus

$$(1.4) \quad \begin{aligned} T &:= \{E \in \mathrm{Coh}(X) \mid R^1\pi_*(G^\vee \otimes E) = 0\} \\ S &:= \{E \in \mathrm{Coh}(X) \mid \pi_*(G^\vee \otimes E) = 0\}, \end{aligned}$$

is a torsion pair (T, S) of $\mathrm{Coh}(X)$, $G \in T$ and

$$(1.5) \quad \mathfrak{C} = \{E \in \mathbf{D}(X) \mid H^i(E) = 0, i \neq -1, 0, H^{-1}(E) \in S, H^0(E) \in T\}.$$

\mathfrak{C} is the heart of a bounded t -structure of $\mathbf{D}(X)$. For $E \in \mathbf{D}(X)$, ${}^p H^i(E) \in \mathfrak{C}$ denotes the i -th cohomology of E with respect to the t -structure. We take a divisor H on X which is the pull-back of an ample divisor on Y . By using a twisted Hilbert polynomial $\chi(G, E(nH))$ of $E \in \mathfrak{C}$, we define the dimension and the torsion

freeness of E , which depend only on the category \mathfrak{C} . We also define a G -twisted semi-stability of $E \in \mathfrak{C}$ as in the Gieseker semi-stability [28, Prop. 1.4.3]. Then we have the following.

Proposition 1.2 ([22], [28, Prop. 1.4.3]). *There is a coarse moduli scheme $M_H^G(v)$ of S -equivalence classes of G -twisted semi-stable objects E with $v(E) = v$.*

Definition 1.3. For $\gamma \in \text{NS}(X)_{\mathbb{Q}}$ such that there is a local projective generator G of \mathfrak{C} with $\gamma = c_1(G)/\text{rk } G$, we also define γ -twisted semi-stability as G -twisted semi-stability.

- (1) We denote the moduli stack of γ -twisted semi-stable objects E with $v(E) = v$ by $\mathcal{M}_H^\gamma(v)$. It is a quotient stack of a scheme by a group action (see the construction in [28, Prop. 1.4.3]). $\mathcal{M}_H^\gamma(v)^s$ denotes the open substack of $\mathcal{M}_H^\gamma(v)$ consisting of γ -twisted stable objects.
- (2) $\mathcal{M}_H^\gamma(v)$ denotes the coarse moduli scheme of S -equivalence classes of γ -twisted semi-stable objects E with $v(E) = v$.

Remark 1.4. If $\mathfrak{C} = \text{Coh}(X)$, then every locally free sheaf is a local projective generator. Hence any $\gamma \in \text{NS}(X)_{\mathbb{Q}}$ is expressed as $\frac{c_1(G)}{\text{rk } G} = \gamma$ for a locally free sheaf G . γ -twisted semi-stability depends on the equivalence class $\gamma \bmod \mathbb{Q}H$ and coincides with the twisted semi-stability of Matsuki and Wentworth [17].

Definition 1.5. By using the slope function $(c_1(E), H)/\text{rk } E$, we also define the μ -semi-stability of a torsion free object E of \mathfrak{C} . $\mathcal{M}_H(v)^{\mu\text{-ss}}$ denotes the moduli stack of μ -semi-stable objects E with $v(E) = v$.

Remark 1.6. (1) μ -semi-stability depends only on the category \mathfrak{C} and H .
(2) $\mathcal{M}_H(v)^{\mu\text{-ss}}$ is bounded.

Lemma 1.7. *Assume that G satisfies*

$$(1.6) \quad \frac{\text{ch}(G)}{\text{rk } G} = e^{\frac{c_1(G)}{\text{rk } G}} + \left(\frac{1}{8}(K_X^2) - \chi(\mathcal{O}_X) \right) \varrho_X.$$

Then

$$(1.7) \quad \frac{\chi(G, E)}{\text{rk } G} = -(e^\beta, v(E)),$$

where

$$(1.8) \quad \beta - \frac{1}{2}K_X = \frac{c_1(G)}{\text{rk } G}.$$

Proof. For $E \in \mathbf{D}(X)$, we have

$$(1.9) \quad \begin{aligned} \chi(e^{\beta - \frac{1}{2}K_X}, E) &= \int_X \left\{ e^{-\beta} e^{\frac{1}{2}K_X} \text{ch}(E) \left(1 - \frac{1}{2}K_X + \chi(\mathcal{O}_X) \varrho_X \right) \right\} \\ &= \int_X e^{-\beta} \text{ch}(E) \left(1 + (\chi(\mathcal{O}_X) - \frac{1}{8}(K_X^2)) \varrho_X \right) \\ &= \int_X (e^{-\beta} \text{ch}(E)) - \text{rk } E \left(\frac{1}{8}(K_X^2) - \chi(\mathcal{O}_X) \right) \\ &= -(e^\beta, v(E)) - \text{rk } E \left(\frac{1}{8}(K_X^2) - \chi(\mathcal{O}_X) \right). \end{aligned}$$

Hence the claim holds. \square

1.3. Stability condition associated to \mathfrak{C} . For the birational map $\pi : X \rightarrow Y$ in 1.2, let H be the pull-back of an ample divisor on Y . For $\beta \in \text{NS}(X)_{\mathbb{Q}}$ in Lemma 1.7 and $\omega \in \mathbb{R}_{>0}H$, we set

$$(1.10) \quad Z_{(\beta, \omega)}(E) := (e^{\beta + \omega \sqrt{-1}}, v(E)), \quad E \in \mathbf{D}(X).$$

For $E \in \mathbf{D}(X)$, we can write $v(E)$ as

$$(1.11) \quad \begin{aligned} v(E) &= e^\beta (r(E) + a_\beta(E) \varrho_X + d_\beta(E)H + D_\beta(E)), \quad D_\beta(E) \in H^\perp \\ &= r(E)e^\beta + a_\beta(E) \varrho_X + d_\beta(E)H + D_\beta(E) + (d_\beta(E)H + D_\beta(E), \beta) \varrho_X, \end{aligned}$$

where $r(E) = \text{rk}(E)$ is the rank of E and

$$(1.12) \quad d_\beta(E) = \frac{(c_1(E) - r(E)\beta, H)}{(H^2)}, \quad a_\beta(E) = -(e^\beta, v(E)).$$

Then we have

$$(1.13) \quad Z_{(\beta, \omega)}(E) = -a_\beta(E) + r(E) \frac{(\omega^2)}{2} + d_\beta(E)(H, \omega) \sqrt{-1}.$$

In appendix, we shall prove the following inequality.

Lemma 1.8 (Lemma 6.2). *For a μ -semi-stable object E of \mathfrak{C} ,*

$$(v(E)^2) - (D_\beta(E)^2) = d_\beta(E)^2(H^2) - 2 \operatorname{rk} E a_\beta(E) \geq 0.$$

Definition 1.9. (1) \mathcal{T}_β is the subcategory of \mathfrak{C} generated by torsion objects and μ -stable objects E with $d_\beta(E) > 0$.

(2) \mathcal{F}_β is the subcategory of \mathfrak{C} generated by μ -stable objects E with $d_\beta(E) \leq 0$.

Then $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ is a torsion pair of \mathfrak{C} .

Definition 1.10. Let $\mathcal{A}_{(\beta, \omega)}$ be the tilting of the torsion pair $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ of $\operatorname{Coh}(X)$:

$$(1.14) \quad \mathcal{A}_{(\beta, \omega)} = \{E \in \mathbf{D}(X) \mid H^i(E) = 0, i \neq -1, 0, H^{-1}(E) \in \mathcal{F}_\beta, H^0(E) \in \mathcal{T}_\beta\}.$$

By Lemma 1.8, $\sigma_{(\beta, \omega)} := (\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ is an example of stability condition. However we do not know whether $\sigma_{(\beta, \omega)}$ satisfies the support property in general. If π is an isomorphism, a blow-up of a smooth point or the minimal resolution of rational double points, then the usual Bogomolov inequality holds and the support property holds (see subsection 2.4).

Remark 1.11. By Lemma 1.7, a torsion free object $E \in \mathfrak{C}$ is $(\beta - \frac{1}{2}K_X)$ -twisted semi-stable with respect to ω if and only if

$$(1.15) \quad \text{(i) } \frac{d_\beta(F)}{\operatorname{rk} F} < \frac{d_\beta(E)}{\operatorname{rk} E} \text{ or (ii) } \frac{d_\beta(F)}{\operatorname{rk} F} = \frac{d_\beta(E)}{\operatorname{rk} E} \text{ and } \frac{a_\beta(F)}{\operatorname{rk} F} \leq \frac{a_\beta(E)}{\operatorname{rk} E}$$

for all non-zero subobject F of E .

Definition 1.12. For a stability condition σ , $\mathcal{M}_\sigma(v)$ denotes the moduli stack of σ -semi-stable objects E with $v(E) = v$. We also denote $\mathcal{M}_{\sigma_{(\beta, \omega)}}(v)$ by $\mathcal{M}_{(\beta, \omega)}(v)$. $\mathcal{M}_{(\beta, \omega)}(v)^s$ denotes the substack of $\mathcal{M}_{(\beta, \omega)}(v)$ consisting of $\sigma_{(\beta, \omega)}$ -stable objects.

2. PERVERSE COHERENT SHEAVES ON A BLOW-UP

2.1. A local projective generator. Let $\pi : X \rightarrow Y$ be the blow-up of a point of Y and C the exceptional divisor on X . For $\beta \in \operatorname{NS}(X)_\mathbb{Q}$, there is an element $G \in K(X)$ such that

$$(2.1) \quad \frac{\operatorname{ch}(G)}{\operatorname{rk} G} = e^{\beta - \frac{1}{2}K_X} + \left(\frac{1}{8}(K_X^2) - \chi(\mathcal{O}_X) \right) \varrho_X.$$

Then

$$(2.2) \quad \frac{\chi(G, E)}{\operatorname{rk} G} = -(e^\beta, v(E))$$

by Lemma 1.7. Assume that $(\beta, C) \notin \frac{1}{2} + \mathbb{Z}$. We take an integer l satisfying $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$. Then there is a locally free sheaf G satisfying (2.1) and

$$(2.3) \quad \operatorname{Ext}^1(G, \mathcal{O}_C(l)) = \operatorname{Hom}(G, \mathcal{O}_C(l-1)) = 0$$

by [28, sect. 2.4] (see also the next paragraph). We set

$$(2.4) \quad \begin{aligned} T &:= \{F \in \operatorname{Coh}(X) \mid R^1\pi_*(G^\vee \otimes F) = 0\} \\ S &:= \{F \in \operatorname{Coh}(X) \mid \pi_*(G^\vee \otimes F) = 0\}. \end{aligned}$$

Then (T, S) is a torsion pair of $\operatorname{Coh}(X)$. Let \mathfrak{C}^β be a category of perverse coherent sheaves associated to (T, S) . Then G is a local projective generator of \mathfrak{C}^β . It is easy to see that $\mathfrak{C}^\beta(lC)$ is the category of perverse coherent sheaves $^{-1}\operatorname{Per}(X/Y)$ defined by Bridgeland [5] and studied in [21], [22].

We set $G_1 := G(lC)$. By (2.3), $G_{1|C}^\vee \cong \mathcal{O}_C^{\oplus k} \oplus \mathcal{O}_C(-1)^{\oplus(r-k)}$, where $(c_1(G_1), C) = r - k$. Then we have an exact sequence

$$(2.5) \quad 0 \rightarrow E \rightarrow G_1^\vee \rightarrow \mathcal{O}_C(-1)^{\oplus(r-k)} \rightarrow 0$$

such that $E|_C \cong \mathcal{O}_C^{\oplus r}$. Since E is the pull-back of a locally free sheaf on Y , $\pi_*(G_1^\vee) \cong \pi_*(E)$ is a locally free sheaf on Y . By taking the dual of (2.5), we have an exact sequence

$$(2.6) \quad 0 \rightarrow G_1 \rightarrow \pi^*(G_2) \rightarrow \mathcal{O}_C^{\oplus(r-k)} \rightarrow 0,$$

where $\pi^*(G_2) = E^\vee$. Thus G_1 is an elementary transform of the pull-back of a locally free sheaf G_2 on Y . Conversely starting from a locally free sheaf G_2 on Y , we can construct a local projective generator G_1 .

2.2. Some properties of perverse coherent sheaves. We recall some results on stable perverse coherent sheaves in [22].

Lemma 2.1. *Assume that $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$, that is, $\mathfrak{C}^\beta(lC) = {}^{-1}\text{Per}(X/Y)$. For a torsion free object E of \mathfrak{C}^β , there is an exact sequence*

$$(2.7) \quad 0 \rightarrow E \rightarrow E' \rightarrow T \rightarrow 0$$

in \mathfrak{C}^β such that T is 0-dimensional, $E'(lC)$ is the pull-back of a locally free sheaf on Y .

Proof. By Lemma 6.1 in Appendix, there is an exact sequence

$$(2.8) \quad 0 \rightarrow E \rightarrow E' \rightarrow T \rightarrow 0$$

such that T is 0-dimensional, E' is torsion free and $\text{Ext}^1(A, E') = 0$ for all 0-dimensional object A of \mathfrak{C}^β . In particular, $\text{Ext}^1(E', k_x) = \text{Ext}^1(k_x, E')^\vee = 0$ for all $x \in X$, which implies E' is locally free by [28, Lem. 1.1.31]. Let A be an irreducible object of \mathfrak{C}^β . Then $A = \mathcal{O}_C(l)$, $A = \mathcal{O}_C(l-1)[1]$ or k_x ($x \in X \setminus C$) (cf. [28, Lem. 1.2.16, Prop. 1.2.23]). Hence $\text{Ext}^1(E', \mathcal{O}_C(l-1)) = \text{Ext}^1(\mathcal{O}_C(l), E')^\vee = 0$. Since $E' \in \mathfrak{C}^\beta$, we also have $\text{Hom}(E', \mathcal{O}_C(l-1)) = 0$. Hence $E'_C \cong \mathcal{O}_C(l)^{\oplus r}$, which implies that $E'(lC)$ is the pull-back of a locally free sheaf on Y . \square

Definition 2.2. Assume that $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$. Since \mathfrak{C}^β depends only on l , we denote this category by \mathfrak{C}^l . We also denote the moduli stack of μ -semi-stable objects by $\mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v)^{\mu\text{-ss}}$ or $\mathcal{M}_H^l(v)^{\mu\text{-ss}}$ in order to indicate the category where the stability is defined (cf. Remark 1.6).

Remark 2.3. Assume that $\gcd(r, (c_1(v), H)) = 1$. Then $E \in \mathfrak{C}$ is β -twisted semi-stable if and only if $\pi_*(E)^{\vee\vee}$ is stable. In particular it is independent of the choice of β satisfying $-\frac{1}{2} + l < (\beta, C) < \frac{1}{2} + l$.

We shall consider the relation with the μ -semi-stability of torsion free sheaves.

Definition 2.4. A torsion free sheaf E is μ -semi-stable with respect to H , if

$$(2.9) \quad \frac{(c_1(E_1), H)}{\text{rk } E_1} \leq \frac{(c_1(E), H)}{\text{rk } E}$$

for any subsheaf $E_1 \neq 0$ of E .

The set of μ -semi-stable sheaves of a fixed Mukai vector is bounded (use [26, Lem. 2.2, sect. 2.3]). If a μ -semi-stable perverse coherent sheaf E is torsion free in $\text{Coh}(X)$, then it is a μ -semi-stable sheaf. By the proof of [22, Prop. 3.37], we have the following lemma.

Lemma 2.5. *For $m \gg 0$, $\mathcal{M}_H^{\beta+mC - \frac{1}{2}K_X}(v)^{\mu\text{-ss}}$ consists of μ -semi-stable torsion free sheaves with respect to H .*

Proposition 2.6. *For $m \gg 0$, $\mathcal{M}_H^{\beta+mC - \frac{1}{2}K_X}(v) = \mathcal{M}_{H-qC}^{\beta - \frac{1}{2}K_X}(v)$, where $q > 0$ is sufficiently small.*

Proof. We set $\beta_m := \beta + mC$. Let $\mathcal{M}_H(v)^{\mu\text{-ss}}$ be the moduli stack of μ -semi-stable torsion free sheaves with respect to H . By Lemma 2.5, we may assume that $\mathcal{M}_H^{\beta_m - \frac{1}{2}K_X}(v) \subset \mathcal{M}_H(v)^{\mu\text{-ss}}$. We choose a sufficiently small $q > 0$ such that $\mathcal{M}_{H-qC}^{\beta - \frac{1}{2}K_X}(v) \subset \mathcal{M}_H(v)^{\mu\text{-ss}}$. We consider the set T of pairs (E_1, E) such that

- (i) E_1 is a subsheaf of a μ -semi-stable sheaf E of $v(E) = v$ with respect to H ,
- (ii) $\frac{(c_1(E_1), H)}{\text{rk } E_1} = \frac{(c_1(E), H)}{\text{rk } E}$ and
- (iii) E/E_1 is torsion free.

Since $(v(E_1)^2) \geq 0$ and $(v(E/E_1)^2) \geq 0$, we see that $\{v(E_1) \mid (E_1, E) \in T\}$ is a finite set. Since

$$(2.10) \quad \begin{aligned} & \frac{\chi(E(-\beta_m + \frac{1}{2}K_X))}{\text{rk } E} - \frac{\chi(E_1(-\beta_m + \frac{1}{2}K_X))}{\text{rk } E_1} \\ &= m \left(\frac{c_1(E)}{\text{rk } E} - \frac{c_1(E_1)}{\text{rk } E_1}, -C \right) + \frac{\chi(E(-\beta + \frac{1}{2}K_X))}{\text{rk } E} - \frac{\chi(E_1(-\beta + \frac{1}{2}K_X))}{\text{rk } E_1}, \end{aligned}$$

we can take a sufficiently large m such that

$$(2.11) \quad \frac{\chi(E_1(p(H - qC) - (\beta - \frac{1}{2}K_X)))}{\text{rk } E_1} \underset{(\geq)}{\leq} \frac{\chi(E(p(H - qC) - (\beta - \frac{1}{2}K_X)))}{\text{rk } E}, \quad p \gg 0$$

if and only if

$$(2.12) \quad \frac{\chi(E_1(-\beta_m + \frac{1}{2}K_X))}{\text{rk } E_1} \underset{(\geq)}{\leq} \frac{\chi(E(-\beta_m + \frac{1}{2}K_X))}{\text{rk } E}$$

for $(E_1, E) \in T$. If E is a $(\beta_m - \frac{1}{2}K_X)$ -twisted semi-stable object of \mathfrak{E}^{β_m} , then for $(E_1, E) \in T$, we take a subsheaf E'_1 of E_1 such that $E'_1 \in \mathfrak{E}^{\beta_m}$ and $E_1/E'_1 \in \mathfrak{E}^{\beta_m}[-1]$. Then E'_1 is a subobject of E such that $\chi(E'_1(-\beta_m + \frac{1}{2}K_X)) \geq \chi(E_1(-\beta_m + \frac{1}{2}K_X))$. Hence

$$(2.13) \quad \frac{\chi(E_1(-\beta_m + \frac{1}{2}K_X))}{\text{rk } E_1} \leq \frac{\chi(E'_1(-\beta_m + \frac{1}{2}K_X))}{\text{rk } E'_1} \leq \frac{\chi(E(-\beta_m + \frac{1}{2}K_X))}{\text{rk } E},$$

which shows

$$(2.14) \quad \frac{\chi(E_1(p(H - qC) - (\beta - \frac{1}{2}K_X)))}{\text{rk } E_1} \leq \frac{\chi(E(p(H - qC) - (\beta - \frac{1}{2}K_X)))}{\text{rk } E}, \quad p \gg 0.$$

Moreover if the equality holds in (2.14), then we see that $E_1 = E'_1$. Hence E is $(\beta - \frac{1}{2}K_X)$ -semi-stable with respect to $H - qC$.

Conversely for $E \in \mathcal{M}_{H-qC}^{\beta-\frac{1}{2}K_X}(v)$, we may assume that $\text{Hom}(E(-(m+l)C), \mathcal{O}_C(-1)) = 0$. Hence E is an object of \mathfrak{E}^{β_m} . Then we see that E is a $(\beta_m - \frac{1}{2}K_X)$ -semi-stable object. \square

Bogomolov inequality for perverse coherent sheaves is a consequence of [22]. There is another proof in [23].

Lemma 2.7 (Bogomolov inequality). *For a μ -semi-stable object E of \mathfrak{E}^β , $(v(E)^2) \geq 0$.*

Proof. Assume that $E \in \mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v)^{\mu\text{-ss}}$. Take an integer such that $-m - \frac{1}{2} < (\beta, C) < -m + \frac{1}{2}$. By the proof of [22, Prop. 3.15], we have an exact sequence

$$(2.15) \quad 0 \rightarrow \mathcal{O}_C(-m-1)^{\oplus n} \rightarrow E \rightarrow E' \rightarrow 0$$

such that $E' \in \mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v')^{\mu\text{-ss}} \cap \mathcal{M}_H^{\beta-\frac{1}{2}K_X+C}(v')^{\mu\text{-ss}}$, $v' + nv(\mathcal{O}_C(-m-1)) = v$ and

$$n \leq \dim \text{Ext}^1(E', \mathcal{O}_C(-m-1)).$$

Since $\chi(E', \mathcal{O}_C(-m-1)) = -\dim \text{Ext}^1(E', \mathcal{O}_C(-m-1))$, we have $(c_1(E'), C) + rm \geq n$. Hence

$$(2.16) \quad \begin{aligned} (v^2) &= (v'^2) + 2n((c_1(v'), C) + (\frac{1}{2} + m)r) - n^2 \\ &\geq (v'^2) + n^2 + nr > (v'^2). \end{aligned}$$

By the induction on m , the claim follows from Lemma 2.5. \square

Lemma 2.8 ([22, Lem. 3.2]). *Let E be a torsion free object of \mathfrak{E}^0 . Then $(c_1(E), C) \geq 0$.*

Proposition 2.9 (cf. [22, Prop. 3.3]). *Assume that $(c_1(v), C) = 0$. Then $\mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v) \cong \mathcal{M}_{H'}^{\beta'-\frac{1}{2}K_Y}(v')$, where $\pi^*(v') = v$, $0 < (\beta - \frac{1}{2}K_X, C) < 1$, $H' \in \text{Amp}(Y)$ satisfies $\pi^*(H') = H$ and β' is the NS(Y)-component of β .*

Proof. Under the assumption, $\pi^*(\pi_*(E)) \rightarrow E$ is an isomorphism by the proof of [22, Prop. 3.3]. Let G be a local projective generator of \mathfrak{E}^0 with $\frac{c_1(G)}{\text{rk } G} = \beta - \frac{1}{2}K_X$. Then there is a locally free sheaf G_0 on Y and an exact sequence

$$0 \rightarrow \pi^*(G_0^\vee) \rightarrow G^\vee \rightarrow \mathcal{O}_C(-1)^{\oplus p} \rightarrow 0,$$

where $p = (c_1(G), C)$. Then $\frac{c_1(G_0)}{\text{rk } G_0} = \beta' - \frac{1}{2}K_Y$ and $G_0^\vee = \mathbf{R}\pi_*(\pi^*(G_0^\vee)) = \mathbf{R}\pi_*(G^\vee)$. By the projection formula, $\pi_*(G^\vee \otimes E) = G_0^\vee \otimes \pi_*(E)$. For an exact sequence

$$(2.17) \quad 0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

of torsion free objects of \mathfrak{E}^0 , we have $(c_1(E_i), C) \geq 0$ for $i = 1, 2$ by Lemma 2.8. Since $(c_1(E), C) = 0$, we have $(c_1(E_i), C) = 0$ ($i = 1, 2$) which implies $E_i \cong \pi^*(\pi_*(E_i))$. Hence the stability coincides. \square

2.3. Moduli spaces of perverse coherent sheaves on a categorical wall. We shall study the case where β satisfies $(\beta, C) = l - \frac{1}{2}$. In this case, $(e^\beta, v(\mathcal{O}_C(l-1))) = 0$ and we cannot consider a category of perverse coherent sheaves. We shall use the same idea in [22, sect. 3.7] to construct a moduli space of semi-stable perverse coherent sheaves.

We take a locally free sheaf G such that $\chi(G, E)/\text{rk } G = -(e^\beta, v(E))$ and $\text{Hom}(G, \mathcal{O}_C(l-1)[i]) = 0$ for all $i \in \mathbb{Z}$.

Definition 2.10. $E \in \mathfrak{E}^{l-1}$ with $\text{rk } E > 0$ is $(\beta - \frac{1}{2}K_X)$ -twisted semi-stable, if

$$(2.18) \quad \chi(G, E_1(nH)) \leq \text{rk } E_1 \frac{\chi(G, E(nH))}{\text{rk } E}$$

for all subobject E_1 of E in \mathfrak{E}^{l-1} . If the inequality is strict for any non-trivial subobject E_1 of E , then E is $(\beta - \frac{1}{2}K_X)$ -twisted stable.

Let $\mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v)$ be the moduli stack of β -twisted semi-stable objects E with $v(E) = v$.

Remark 2.11. If $E_1 \in \mathfrak{C}^{l-1}$ is a subobject of $E \in \mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v)$ with $\text{rk } E_1 = 0$, then $\chi(G, E_1(nH)) = 0$ for all n by (2.18). Then $E_1 \cong \mathcal{O}_C(l-1)^{\oplus k}$. In particular we see $E \in \text{Coh}(X)$ by $\chi(G, (H^{-1}(E)[1])(nH)) = 0$.

Remark 2.12. If E is β -twisted stable, then $\text{Hom}(E, \mathcal{O}_C(l-1)) = \text{Hom}(\mathcal{O}_C(l-1), E) = 0$. In particular $E \in \mathfrak{C}^l$. Moreover E is a torsion free object of \mathfrak{C}^l and \mathfrak{C}^{l-1} . We also have an exact sequence

$$(2.19) \quad 0 \rightarrow \mathcal{O}_C(-1)^{\oplus N} \rightarrow \pi^*(\pi_*(E(lC))) \rightarrow E(lC) \rightarrow 0.$$

Moreover $\text{Hom}(\mathcal{O}_C(-1), \pi^*(\pi_*(E(lC)))) \cong \mathbb{C}^{\oplus N}$. Hence E is determined by $\pi_*(E(lC))$.

We take $\beta_- \in \text{NS}(X)_{\mathbb{Q}}$ which is sufficiently close to β and $l - \frac{3}{2} < (\beta_-, C) < l - \frac{1}{2}$, and let G_- be a local projective generator of \mathfrak{C}^{l-1} with $\beta_- - \frac{1}{2}K_X = \frac{c_1(G_-)}{\text{rk } G_-}$. For $E \in \mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v)$, we have the Harder-Narasimhan filtration with respect to G_- :

$$(2.20) \quad 0 \subset F_0 \subset F_1 \subset \dots \subset F_s = E.$$

Thus $E_i := F_i/F_{i-1}$ ($i \geq 0$) satisfy the following.

(i) $F_0 = \mathcal{O}_C(l-1)^{\oplus k}$ and E_i ($i \geq 1$) are $(\beta_- - \frac{1}{2}K_X)$ -semi-stable objects,

(ii)

$$\frac{\chi(G_-, E_1(nH))}{\text{rk } E_1} > \frac{\chi(G_-, E_2(nH))}{\text{rk } E_2} > \dots > \frac{\chi(G_-, E_s(nH))}{\text{rk } E_s}, \quad n \gg 0$$

and

(iii)

$$\frac{\chi(G, E_1(nH))}{\text{rk } E_1} = \frac{\chi(G, E_2(nH))}{\text{rk } E_2} = \dots = \frac{\chi(G, E_s(nH))}{\text{rk } E_s}, \quad n \gg 0.$$

If E is β -twisted stable, then $F_0 = 0$ and $s = 1$. Moreover E is $(\beta_- - \frac{1}{2}K_X)$ -twisted stable.

We take β_+ which is sufficiently close to β and $\beta_+ + \beta_- = 2\beta$. Then $l - \frac{1}{2} < (\beta_+, C) < l + \frac{1}{2}$. Let G_+ be a local projective generator of \mathfrak{C}^l with $\beta_+ - \frac{1}{2}K_X = \frac{c_1(G_+)}{\text{rk } G_+}$. Then we have a Harder-Narasimhan type filtration with respect to G_+ :

$$(2.21) \quad 0 \subset F_1 \subset F_2 \subset \dots \subset F_t = E.$$

Thus $E_i := F_i/F_{i-1}$ ($i = 1, 2, \dots, t$) satisfy

(i) E_i ($i = 1, 2, \dots, t-1$) are $(\beta_+ - \frac{1}{2}K_X)$ -twisted semi-stable objects of \mathfrak{C}^l and $E_t = \mathcal{O}_C(l-1)^{\oplus k}$,

(ii)

$$\frac{\chi(G_+, E_1(nH))}{\text{rk } E_1} > \frac{\chi(G_+, E_2(nH))}{\text{rk } E_2} > \dots > \frac{\chi(G_+, E_{t-1}(nH))}{\text{rk } E_{t-1}}, \quad n \gg 0$$

and

(iii)

$$\frac{\chi(G, E_1(nH))}{\text{rk } E_1} = \frac{\chi(G, E_2(nH))}{\text{rk } E_2} = \dots = \frac{\chi(G, E_{t-1}(nH))}{\text{rk } E_{t-1}}, \quad n \gg 0.$$

Remark 2.13.

$$E_t = \text{im}(E \rightarrow \mathcal{O}_C(l-1) \otimes \text{Hom}(E, \mathcal{O}_C(l-1))^\vee).$$

If E is $(\beta - \frac{1}{2}K_X)$ -twisted stable, then $F_t/F_{t-1} = 0$ and $t = 2$. Moreover E is $(\beta_+ - \frac{1}{2}K_X)$ -twisted stable.

Remark 2.14. Since $\mathcal{O}_C(l-1) \in \mathfrak{C}^{\beta-}$ is an irreducible object and $\mathcal{O}_C(l-1)[1]$ is an irreducible object of $\mathfrak{C}^{\beta+}$, we also set $\mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v) = \mathcal{M}_H^{\beta\pm-\frac{1}{2}K_X}(v) = \{\mathcal{O}_C(l-1)^{\oplus n}\}$ for $v = nv(\mathcal{O}_C(l-1))$ ($n > 0$).

Lemma 2.15. Assume that $\text{rk } v > 0$. Then $\mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v)^s = \mathcal{M}_H^{\beta--\frac{1}{2}K_X}(v)^s \cap \mathcal{M}_H^{\beta+-\frac{1}{2}K_X}(v)^s$. The same claim also holds for $v = nv(\mathcal{O}_C(l-1))$.

Proof. We already know that

$$\mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v)^s \subset \mathcal{M}_H^{\beta--\frac{1}{2}K_X}(v)^s \cap \mathcal{M}_H^{\beta+-\frac{1}{2}K_X}(v)^s.$$

Assume that $E \in \mathcal{M}_H^{\beta--\frac{1}{2}K_X}(v)^s \cap \mathcal{M}_H^{\beta+-\frac{1}{2}K_X}(v)^s$. Then $E \in \mathfrak{C}^l$ and $\text{Hom}(\mathcal{O}_C(l-1), E) = 0$. If E is not $(\beta - \frac{1}{2}K_X)$ -twisted stable, then there is a subobject E_1 of E in \mathfrak{C}^{l-1} with

$$(2.22) \quad \chi(G, E_1(nH)) \geq \text{rk } E_1 \frac{\chi(G, E(nH))}{\text{rk } E}.$$

We have an exact sequence

$$0 \rightarrow E'_1 \rightarrow E_1 \rightarrow \mathcal{O}_C(l-1)^{\oplus q} \rightarrow 0$$

such that $E'_1 \in \mathfrak{C}^l$. Then E'_1 is a subobject of E in \mathfrak{C}^l and \mathfrak{C}^{l-1} with $\text{rk } E'_1 = \text{rk } E_1$ and $\chi(G, E'_1(nH)) = \chi(G, E_1(nH))$. By $\text{Hom}(\mathcal{O}_C(l-1), E) = 0$, $E'_1 \neq 0$. Then the G_\pm -twisted stability implies

$$(2.23) \quad \chi(G_\pm, E'_1(nH)) < \text{rk } E'_1 \frac{\chi(G_\pm, E(nH))}{\text{rk } E},$$

which shows

$$(2.24) \quad \chi(G, E_1(nH)) < \text{rk } E_1 \frac{\chi(G, E(nH))}{\text{rk } E}.$$

Therefore E is $(\beta - \frac{1}{2}K_X)$ -twisted stable. \square

We shall explain a construction of the moduli space. There is a locally free sheaf G_0 on Y such that $\pi^*(G_0)(-lC) \cong G$. Then $E \in \mathfrak{C}^{l-1}$ is $(\beta - \frac{1}{2}K_X)$ -twisted semi-stable if and only if $\mathbf{R}\pi_*(E(lC))$ is G_0 -twisted semi-stable. Moreover if E is $(\beta - \frac{1}{2}K_X)$ -twisted stable, then $\pi_*(E(lC))$ is $(\beta' - \frac{1}{2}K_Y)$ -twisted stable, where $\beta' - \frac{1}{2}K_Y = \frac{c_1(G_0)}{\text{rk } G_0}$. We have a morphism

$$(2.25) \quad \phi : \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v) \rightarrow M_{H'}^{\beta' - \frac{1}{2}K_Y}(w),$$

where $w = v(\pi_*(E(lC)))$. If E is S -equivalent to $\oplus_{i=0}^s E_i$ such that $E_0 = \mathcal{O}_C(l-1)^{\oplus p}$ and E_i ($i > 0$) are $(\beta - \frac{1}{2}K_X)$ -twisted stable objects with $\text{rk } E_i > 0$. Then $\pi_*(E(lC))$ is S -equivalent to $\oplus_{i=1}^s \pi_*(E_i(lC))$ and $\pi_*(E_i(lC))$ are $(\beta' - \frac{1}{2}K_Y)$ -twisted stable. By Remark 2.12, $\oplus_{i=1}^s E_i$ is uniquely determined by $\pi_*(E(lC))$. Since $pC = c_1(E) - \sum_{i=1}^s c_1(E_i)$, E_0 is also uniquely determined.

We set $k := r((\delta, C) - l)$ and

$$(2.26) \quad M_{H'}^{\beta' - \frac{1}{2}K_Y}(w, k) := \{F \in M_{H'}^{\beta' - \frac{1}{2}K_Y}(w) \mid \dim \text{Hom}(\mathcal{O}_C(-1), \pi^*(F)) \geq k\}.$$

We shall introduce a scheme structure on this Brill-Noether locus as in [22, Prop. 3.31].

Lemma 2.16. *The image of ϕ is $M_{H'}^{\beta' - \frac{1}{2}K_Y}(w, k)$.*

Proof. For $E \in M_H^{\beta - \frac{1}{2}K_X}(v)$, we have an exact sequence

$$(2.27) \quad 0 \rightarrow E'(lC) \rightarrow E(lC) \rightarrow \mathcal{O}_C(-1)^{\oplus q} \rightarrow 0$$

where $E' \in \mathfrak{C}^l$. Then $\pi_*(E(lC)) = \pi_*(E'(lC))$ and

$$(2.28) \quad \begin{aligned} \dim \text{Hom}(\mathcal{O}_C(-1), \pi^*(\pi_*(E'(lC)))) &\geq \dim \text{Ext}^1(E'(lC), \mathcal{O}_C(-1)) \\ &= -\chi(E'(lC), \mathcal{O}_C(-1)) = q + k. \end{aligned}$$

Hence $\phi(E) \in M_{H'}^{\beta' - \frac{1}{2}K_Y}(w, k)$. Conversely for $F \in M_{H'}^{\beta' - \frac{1}{2}K_Y}(w, k)$, we take a k -dimensional subspace V of $\text{Hom}(\mathcal{O}_C(-1), \pi^*(F))$. Then $E(lC) := \text{coker}(\mathcal{O}_C(-1) \otimes V \rightarrow \pi^*(F)) \in \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v)$ and $\phi(E) = F$. \square

Definition 2.17. We define the moduli scheme of the S -equivalence classes of $(\beta - \frac{1}{2}K_X)$ -twisted semi-stable objects by $M_H^{\beta - \frac{1}{2}K_X}(v) := M_{H'}^{\beta' - \frac{1}{2}K_Y}(w, k)$.

Since $\mathcal{M}_H^{\beta \pm - \frac{1}{2}K_X}(v) \subset \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v)$, we have the following.

Proposition 2.18. *There are projective morphisms $M_H^{\beta \pm - \frac{1}{2}K_X}(v) \rightarrow M_H^{\beta - \frac{1}{2}K_X}(v)$.*

2.4. Support property. Let us recall Bogomolov type inequality and the support property in [23], [24]. In these paper, Toda states results under the assumption $\beta = 0$. Obviously the same results hold for general cases.

Lemma 2.19 ([23, Lem. 3.20]). *There is a constant C_ω depending only on the class $\omega \in C^+(X)$ such that*

$$(2.29) \quad (c_1(E)^2)(\omega^2) + C_\omega(c_1(E), \omega)^2 \geq 0$$

for all purely 1-dimensional objects $E \in \mathfrak{C}^\beta$.

Proof. If ω is ample, then it is nothing but [23, Lem. 3.20]. So we assume that $\omega \in \pi^*(\text{Amp}(X))$. If $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$, then we have $\mathfrak{C}^\beta(lC) = {}^{-1}\text{Per}(X/Y)$ and $c_1(E) = c_1(E(lC))$. Hence the claim also follows from [23, Lem. 3.20]. \square

Then the Bogomolov type inequality [23, Cor. 3.24] holds.

Lemma 2.20. *For any $\sigma_{(\beta, \omega)}$ -stable object $E \in \mathcal{A}_{(\beta, \omega)}$,*

$$(2.30) \quad (v(E)^2) + C_\omega \frac{(c_1(E(-\beta)), \omega)^2}{(\omega^2)} \geq -1.$$

As in [23, sect. 3.7], there is a constant A depending on C_ω , (ω^2) , $\frac{1}{a_\beta(\mathcal{O}_C(l))}$ and $\frac{1}{1-a_\beta(\mathcal{O}_C(l))}$ such that

$$(2.31) \quad \frac{\max\{|r|, |a|, |d(\omega^2)|, \sqrt{-(\eta^2)}\}}{|Z_{(\beta, \omega)}(E)|} \leq A$$

for all $\sigma_{(\beta, \omega)}$ -stable objects E , where $v(E) = e^\beta(r + d\omega + \eta + a\rho_X)$, $\eta \in \omega^\perp$ and $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$. More precisely, if $E \neq \mathcal{O}_C(l), \mathcal{O}_C(l-1)[1]$, then we have a bound A depending only on C_ω and (ω^2) as in [23, sect. 3.7]. If $E = \mathcal{O}_C(l), \mathcal{O}_C(l-1)[1]$, then $\frac{1}{a_\beta(\mathcal{O}_C(l))}$ and $\frac{1}{1-a_\beta(\mathcal{O}_C(l))}$ appear.

In particular, we have the following support property.

Lemma 2.21 ([23, Prop. 3.13]). *Let B be a compact subset of $\text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R}$. Then there is a constant C_B such that for $(\beta, \omega) \in B$ satisfying $\beta \in \text{NS}(X)_\mathbb{Q}$, $\omega \in \mathbb{R}_{>0}H$, $H \in \text{Amp}(X)_\mathbb{Q}$ and $\sigma_{(\beta, \omega)}$ -stable object E ,*

$$(2.32) \quad \frac{\|v(E)\|}{|Z_{(\beta, \omega)}(E)|} \leq C_B,$$

where $\|v(E)\|$ is a fixed norm on $H^*(X, \mathbb{R})_{\text{alg}}$. Assume that B is a compact subset of $\text{NS}(X)_\mathbb{R} \times \pi^*(\text{Amp}(Y)_\mathbb{R})$ such that $(\beta, \omega) \in B$ satisfies $(\beta, C) \notin \frac{1}{2} + \mathbb{Z}$. Then (2.32) also holds.

Let (β, ω) satisfy the conditions in Lemma 2.21. By the support property, we have a wall/chamber structure in a neighborhood U of $\sigma_{(\beta, \omega)}$. Thus

$$(2.33) \quad \{\sigma \in U \mid Z_\sigma(E)/Z_\sigma(E_1) \in \mathbb{R}_{>0}, E_1 \subset E \text{ in } \mathcal{A}_\sigma, E, E_1 \text{ are } \sigma\text{-semi-stable}, v(E) = v\}$$

is the set of walls for v and a connected component of the complement is a chamber in U .

Remark 2.22. Assume that $\text{NS}(X) = \mathbb{Z}H$. Then the Bogomolov inequality holds for any semi-stable object of $\mathcal{A}_{(\beta, \omega)}$. As in [19, Prop. 5.7], if v_1 defines a wall, then we have $(v_1^2) \geq 0$, $(v_2^2) \geq 0$ and $(v_1, v_2) > 0$, where $v_2 = v - v_1$. Since we don't know the sufficient condition for the existence of stable objects E_i with $v(E_i) = v_i$, the condition is only a necessary condition.

We set

$$(2.34) \quad \begin{aligned} \mathcal{K}(X) &:= \{(\beta, \omega) \mid \beta \in \text{NS}(X)_\mathbb{R}, \omega \in \text{Amp}(X)_\mathbb{R}\} \\ \partial\mathcal{K}(X) &:= \{(\beta, \omega) \mid \beta \in \text{NS}(X)_\mathbb{R}, \omega \in \pi^*(\text{Amp}(Y)_\mathbb{R}), (\beta, C) \notin \frac{1}{2} + \mathbb{Z}\} \\ \overline{\mathcal{K}}(X) &:= \mathcal{K}(X) \cup \partial\mathcal{K}(X). \end{aligned}$$

We have a natural embedding $\overline{\mathcal{K}}(X) \rightarrow \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$.

Proposition 2.23 ([24]). *We have a map $\mathfrak{s} : \overline{\mathcal{K}}(X) \rightarrow \text{Stab}(X)$ such that $Z_{\mathfrak{s}(\beta, \omega)} = e^{\beta + \sqrt{-1}\omega}$ and $\mathfrak{s}(\beta, \omega) = \sigma_{(\beta, \omega)}$ if $\beta \in \text{NS}(X)_\mathbb{Q}$ and $\omega \in \mathbb{R}_{>0}H$, $H \in \text{NS}(X)$.*

For the proof of this result, we also need the following. Then the same proof of [23] works.

Lemma 2.24 ([23, Prop. 3.14]). *Assume that $(\beta_0, \omega_0) \in \partial\mathcal{K}(X)$. Let U be an open neighborhood of $\sigma_0 := \sigma_{(\beta_0, \omega_0)}$ in $\text{Stab}(X)$. We set $V := \{\sigma \mid \Pi(\sigma) = e^{\beta + \sqrt{-1}\omega}, (\omega, C) \geq 0\}$. Then there is an open neighborhood U' of σ_0 such that $\mathcal{M}_\sigma(\rho_X) = \{k_x \mid x \in X\}$ for all $\sigma \in U' \cap V$.*

Proof. For $E \in \mathcal{M}_{\sigma_0}(\rho_X)$, $\phi_{\sigma_0}(E) = 1$. We first classify σ_0 -stable objects E with $\phi_{\sigma_0}(E) = 1$. Since $\phi_{\sigma_0}(^pH^{-1}(E)[1]), \phi_{\sigma_0}(^pH^0(E)) \leq 1$, $\phi_{\sigma_0}(^pH^{-1}(E)[1]) = \phi_{\sigma_0}(^pH^0(E)) = 1$. Hence $^pH^0(E) = 0$ or $^pH^{-1}(E) = 0$. In the first case, $0 = -\text{rk } E = \text{rk } ^pH^{-1}(E) > 0$. Hence this case does not occur. For the second case, $E = ^pH^0(E)$ satisfies $\text{rk } ^pH^0(E) = (c_1(E), \omega) = 0$. Hence E is a 0-dimensional object of \mathfrak{E}^β . If $\text{Supp}(E) \subset X \setminus C$, then $E \cong k_x$, $x \in X \setminus C$. Assume that $\text{Supp}(E) \subset C$. We may assume $\mathfrak{E}^\beta(lC) = {}^{-1}\text{Per}(X/Y)$. By the classification of 0-dimensional objects, E is generated by $\mathcal{O}_C(l)$ and $\mathcal{O}_C(l-1)[1]$. Thus $E = \mathcal{O}_C(l)$ or $E = \mathcal{O}_C(l-1)[1]$. We note that $\rho_X = n_1v(\mathcal{O}_C(l)) + n_2v(\mathcal{O}_C(l-1)[1])$ if and only if $n_1 = n_2 = 1$. It is easy to see that $\mathcal{M}_{\sigma_0}(\rho_X)$ consists of E such that

(i) E is a non-trivial extensions

$$0 \rightarrow \mathcal{O}_C(l) \rightarrow E \rightarrow \mathcal{O}_C(l-1)[1] \rightarrow 0$$

or

(ii)

$$0 \rightarrow \mathcal{O}_C(l-1)[1] \rightarrow E \rightarrow \mathcal{O}_C(l) \rightarrow 0$$

or

(iii) $E = \mathcal{O}_C(l) \oplus \mathcal{O}_C(l-1)[1]$ or

(iv) $E = k_x$ ($x \in X \setminus C$).

We may assume that $\mathcal{O}_C(l)$ and $\mathcal{O}_C(l-1)[1]$ are σ -stable for all $\sigma \in U'$. If $\sigma \in V$, then $\phi_\sigma(\mathcal{O}_C(l)) < 1 < \phi_\sigma(\mathcal{O}_C(l-1)[1])$. Hence if E is σ -semi-stable for $\sigma \in U' \cap V$, then E is an extension of the first type. Therefore $E \cong k_x$, $x \in C$. \square

3. STRUCTURES OF WALLS AND CHAMBERS

Let X be a smooth projective surface. In this section, we shall study the structure of walls and chambers. Let $P^+(v)_\mathbb{R} \subset v^\perp$ be the positive cone in v^\perp and set $C^+(v) := P^+(v)_\mathbb{R}/\mathbb{R}_{>0}$. We shall construct a map

$$\xi : \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R} \rightarrow C^+(v)$$

and study its property (Proposition 3.2, Corollary 3.5).

Let $v = r + c_1(v) + a\rho_X$ be an element of $v(K(X)) \subset H^*(X, \mathbb{Q})_{\text{alg}}$ with $r > 0$. From now on, we set

$$(3.1) \quad \delta := \frac{c_1(v)}{r}.$$

Then we have

$$v = re^\delta - \frac{(v^2)}{2r}\rho_X.$$

As in [29], we set

$$(3.2) \quad \begin{aligned} \xi(\beta, \omega) &:= \xi(\beta, \omega, 1)/r \\ &= \left(\frac{(\omega^2) - ((\beta - \delta)^2)}{2} + \frac{(v^2)}{2r^2} \right) (\omega + (\omega, \delta)\rho_X) \\ &\quad + (\beta - \delta, \omega)(\beta - \delta + (\beta - \delta, \delta)\rho_X) + (\beta - \delta, \omega) \left(e^\delta + \frac{(v^2)}{2r^2}\rho_X \right) \in C^+(v) \end{aligned}$$

for $(\beta, \omega) \in \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$. We have

$$\xi(\beta, \omega) = \text{Im} \frac{e^{\beta + \sqrt{-1}\omega}}{Z_{(\beta, \omega)}(v)} \in C^+(v).$$

For $v_1 \in H^*(X, \mathbb{Q})_{\text{alg}}$, $Z_{(\beta, \omega)}(v_1) \in \mathbb{R}Z_{(\beta, \omega)}(v)$ if and only if $\xi(\beta, \omega) \in v_1^\perp$.

Remark 3.1. For $u \in v^\perp$ with $(u^2) > 0$, $(\beta, \omega) \in \xi^{-1}(u)$ if and only if $Z_{(\beta, \omega)}(x) \in \mathbb{R}Z_{(\beta, \omega)}(v)$ for all $x \in u^\perp$.

The main result of this subsection is the following.

Proposition 3.2. $\xi : \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R} \rightarrow C^+(v)$ is a regular map whose fibers are connected. In particular, $\xi^{-1}(U)$ is connected if U is connected. Moreover the restriction of a fiber to $\mathcal{K}(X)$ is also connected.

Before proving this proposition, we shall give consequences of the proposition. We consider the chamber structure in $\text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R}$ and the map

$$(3.3) \quad \xi : \text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R} \rightarrow C^+(v).$$

By the identification $\overline{\mathcal{K}}(X) \cong \mathfrak{s}(\overline{\mathcal{K}}(X))$, we have a map

$$\tilde{\xi} : \mathfrak{s}(\overline{\mathcal{K}}(X)) \rightarrow C^+(v).$$

For $E \in \mathbf{D}(X)$ with $v(E) = v$, assume that E is σ -semi-stable and is S -equivalent to $\oplus_i E_i$, where $\sigma \in D := \tilde{\xi}^{-1}(u)$ and E_i are σ -stable with the same phase. Then there is an open neighborhood U of σ in D such that E_i are σ' -stable for $\sigma' \in U$. Since $\phi_{\sigma'}(E_i), \phi_{\sigma'}(E_j)$ are continuous over U and $\phi_{\sigma'}(E_i) - \phi_{\sigma'}(E_j) \pmod{\mathbb{Z}}$ is constant over D , $\phi_{\sigma'}(E_i) = \phi_{\sigma'}(E_j)$ for all i, j . Hence E is a σ' -semi-stable object which is S -equivalent to $\oplus_i E_i$. Therefore semi-stability is an open condition. If E is not σ' -semi-stable for $\sigma' \in D$, then in a neighborhood of σ' , E is not semi-stable. Assume that D is connected. Then E is σ' -semi-stable for all $\sigma' \in D$. Assume that E is σ -semi-stable and is S -equivalent to $\oplus_i E_i$, where E_i are σ -stable with the same phase. Then we also see that E_i are σ' -stable for all $\sigma' \in D$. Thus the S -equivalence class of E is independent of $\sigma \in D$. In particular, we have the following.

Lemma 3.3. Let $\sigma_0 \in \mathfrak{s}(\overline{\mathcal{K}}(X))$ and assume that $\tilde{\xi}^{-1}(\tilde{\xi}(\sigma_0))$ is connected. If σ_0 is on a wall, then every $\sigma \in \tilde{\xi}^{-1}(\tilde{\xi}(\sigma_0))$ is on the same wall. In particular, if $\sigma_0 \in \mathfrak{s}(\mathcal{K}(X))$, then the claim holds.

Lemma 3.4. Let $u \in P^+(v)_\mathbb{R}$ satisfy $\text{rk } u \neq 0$ or $u = e^\delta \lambda$ ($\lambda \in \text{Amp}(X)_\mathbb{R}$). Then the set of walls in $P^+(v)_\mathbb{R}$ is finite in a neighborhood of u .

Proof. Let V be a compact small neighborhood of x of $C^+(v)$. Then there is a compact subset \tilde{V} of $\text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R}$ such that $\xi(\tilde{V}) = V$ by Proposition 3.2. Then there are finitely many walls intersecting \tilde{V} . By Lemma 3.3, the set of walls in $P^+(v)_\mathbb{R}$ is locally finite. \square

Corollary 3.5. Let W be the set of vectors defining walls with respect to v . Let U be a connected component of $C^+(v) \setminus \cup_{u \in W} u^\perp$. Then $\xi^{-1}(U) \cap \text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R}$ is a chamber for v .

Proof. A chamber is a connected component of $\text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R} \setminus \cup_{u \in W} W_u$ and $(\beta, \omega) \in W_u$ if and only if $\xi(\beta, \omega) \in u^\perp$. By Proposition 3.2, $\xi^{-1}(U)$ is connected. Hence it is a chamber. \square

Corollary 3.5 implies that we can study the wall crossing behavior by looking at linear walls in $C^+(v)$.

Proof of Proposition 3.2.

We set

$$(3.4) \quad u := \zeta + (\zeta, \delta) \varrho_X + y \left(e^\delta + \frac{(v^2)}{2r^2} \varrho_X \right).$$

Then $u \in C^+(v)$ if and only if $(u^2) > 0$ and $(u, H + (H, \delta) \varrho_X) = (\zeta, H) > 0$. Assume that $y \neq 0$. For $H \in P^+(X)_\mathbb{R}$, we have a decomposition

$$(3.5) \quad \zeta = \frac{(\zeta, H)}{(H^2)} H + D, \quad D = \zeta - \frac{(\zeta, H)}{(H^2)} H \in H^\perp.$$

Then

$$(3.6) \quad \frac{\xi(\delta + sH + D', tH)}{(sH + D', tH)} = \frac{\frac{(H^2)}{2}(t^2 + s^2) + \frac{(v^2)}{2r^2} - \frac{(D'^2)}{2}}{s(H^2)} (H + (H, \delta) \varrho_X) + D' + (D', \delta) \varrho_X + \left(e^\delta + \frac{(v^2)}{2r^2} \varrho_X \right),$$

where $D' \in H^\perp$. Hence $\frac{\xi(\delta + sH + D', tH)}{(sH + D', tH)} = \frac{u}{y}$ if and only if

$$(3.7) \quad D' = \frac{D}{y}, \quad \frac{\frac{(H^2)}{2}(t^2 + s^2) + \frac{(v^2)}{2r^2} - \frac{(D^2)}{2y^2}}{s(H^2)} = \frac{(\zeta, H)}{y(H^2)}.$$

Since

$$(3.8) \quad \begin{aligned} & \frac{(H^2)}{2}(t^2 + s^2) + \frac{(v^2)}{2r^2} - \frac{(D^2)}{2y^2} - s \frac{(\zeta, H)}{y} \\ &= \frac{(H^2)}{2}(t^2 + s^2) + \frac{(v^2)}{2r^2} - \frac{1}{2y^2} \left((\zeta^2) - \frac{(\zeta, H)^2}{(H^2)} \right) - s \frac{(\zeta, H)}{y} \\ &= \frac{(H^2)}{2} t^2 + \frac{(H^2)}{2} \left(s - \frac{(\zeta, H)}{y(H^2)} \right)^2 + \frac{(v^2)}{2r^2} - \frac{(\zeta^2)}{2y^2} \\ &= \frac{(H^2)}{2} t^2 + \frac{(H^2)}{2} \left(s - \frac{(\zeta, H)}{y(H^2)} \right)^2 - \frac{(u^2)}{2y^2}, \end{aligned}$$

$\xi^{-1}(u)$ is parameterized by H and the circle

$$(3.9) \quad \frac{(H^2)}{2} t^2 + \frac{(H^2)}{2} \left(s - \frac{(\zeta, H)}{y(H^2)} \right)^2 = \frac{(u^2)}{2y^2}.$$

For the general case, we apply Proposition 3.9 to isometries $(r, \xi, a) \mapsto e^\eta(r, \xi, a)$ and $(r, \xi, a) \mapsto (a, -\xi, r)$ in order to reduce to the case $y \neq 0$. Thus ξ is a regular map as a C^∞ -map. Since we are restricted to $\text{Amp}(X)_\mathbb{R}$, for the connectedness of the fiber of ξ , we need to describe it directly. For $u = H + (H, \delta) \varrho_X$,

$$\xi^{-1}(u) = \{(\beta, \omega) \mid \beta = \delta + D, \omega \in \mathbb{R}_{>0} H, D \in H^\perp\}.$$

So it is connected. □

Remark 3.6. In order to clarify the dependence of $\xi(\beta, \omega)$ on v , we set

$$\xi_v(\beta, \omega) := \text{Im} \frac{e^{\beta + \sqrt{-1}\omega}}{Z_{(\beta, \omega)}(v)}.$$

Then

$$\xi_v^{-1}(u) = \xi_w^{-1}(u)$$

for $u \in v^\perp \cap w^\perp$.

Indeed for $u \in v^\perp$ with $\text{rk } u \neq 0$, by using the expression of u in (3.4), $c_1(u) = \zeta + y\delta$ and (3.9), we see that $\xi_v(\beta, \omega) = u$ in $C^+(v)$ if and only if

$$\frac{(\omega^2)}{2} + \left(\frac{(y\beta - c_1(u), \omega)}{y(\omega^2)} \right)^2 \frac{(\omega^2)}{2} = \frac{(u^2)}{2y^2}.$$

If $\text{rk } u = 0$, then $(c_1(v) - r\beta, \omega) = 0$ and $\omega \in \mathbb{R}_{>0} c_1(u)$. Hence it does not depend on the choice of v . The same claim also holds if $r = 0$.

Remark 3.7. By [16], the walls form nested circles, if we fixed $\mathbb{R}_{>0}\omega$. By our proof of Proposition 3.2, the circles are the fibers of ξ .

We set

$$(3.10) \quad \begin{aligned} x_0 &:= e^\delta H = H + (H, \delta) \varrho_X, \\ x_1 &:= -e^\delta \left(D + \left(1 + \frac{(v^2)}{2r^2} \varrho_X \right) \right) = - \left(D + (D, \delta) \varrho_X + \left(e^\delta + \frac{(v^2)}{2r^2} \varrho_X \right) \right). \end{aligned}$$

Let L be the line in $\mathbb{P}(v^\perp)$ passing through x_0 and x_1 . Then $\xi(\delta + sH + D, tH) \in L$ for all (s, t) . The image of the unbounded chamber in (s, t) with $s < 0$ and $t > 0$ is the interior of a segment connecting x_0 and $x_0 + \epsilon x_1$ ($0 < \epsilon \ll 1$).

Lemma 3.8. *Let H be an ample divisor on X .*

- (1) $\mathcal{M}_{(\delta+sH+D, tH)}(v) = \mathcal{M}_H^{\delta+D-\frac{1}{2}K_X}(v)$ if $-1 \ll s < 0$.
- (2) If (β, ω) is general, then there is $\beta' \in \text{NS}(X)_\mathbb{Q}$ with $\mathcal{M}_{(\beta, \omega)}(v) \cong \mathcal{M}_{(\beta', tH)}(v)$. In particular, $\mathcal{M}_\omega^{\beta-\frac{1}{2}K_X}(v) \cong \mathcal{M}_{(\beta', tH)}(v)$.

Proof. (1) is a consequence of Proposition 6.11. (2) For a chamber in $C^+(v)$ containing $\xi(\beta, \omega)$, we take a vector u and write it as in (3.4) with (3.5). Then there is (s, t) such that $\xi(\delta + sH + D/y, tH) = u$. For $\beta' := \delta + sH + D/y$, $\mathcal{M}_{(\beta, \omega)}(v) \cong \mathcal{M}_{(\beta', tH)}(v)$. \square

We shall remark the behavior of $\xi(\beta, tH)$ under an isometry of $H^*(X, \mathbb{Q})_{\text{alg}}$. Let

$$\Phi : H^*(X, \mathbb{Q})_{\text{alg}} \rightarrow H^*(X, \mathbb{Q})_{\text{alg}}$$

be an isomery. Then Φ induces an isomorphism $\Phi^+ : H^*(X, \mathbb{Q})_{\text{alg}}^+ \rightarrow H^*(X, \mathbb{Q})_{\text{alg}}^+$ of positive 2-plane $H^*(X, \mathbb{Q})_{\text{alg}}^+ = \mathbb{Q}(1 - \varrho_X) + \mathbb{Q}H$. Assume that Φ is an isometry such that $\Phi(r_1 e^\gamma) = \varrho_{X_1}$, $\Phi(\varrho_X) = r_1 e^{\gamma'}$ and Φ^+ preserves the orientation of $H^*(X, \mathbb{Q})_{\text{alg}}^+$. Then we can describe the action as

$$\Phi(re^\gamma + a\varrho_X + \xi + (\xi, \gamma)\varrho_X) = \frac{r}{r_1} \varrho_X + r_1 a e^{\gamma'} - \frac{r_1}{|r_1|} (\widehat{\xi} + (\widehat{\xi}, \gamma') \varrho_X),$$

where $\xi \in \text{NS}(X)_\mathbb{Q}$ and $\widehat{\xi} := \frac{r_1}{|r_1|} c_1(\Phi(\xi + (\xi, \gamma)\varrho_X)) \in \text{NS}(X)_\mathbb{Q}$. We note that ξ belongs to the positive cone if and only if $\widehat{\xi}$ belongs to the positive cone. For $(\beta, \omega) \in \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$, we set

$$(3.11) \quad \begin{aligned} \widetilde{\omega} &:= -\frac{1}{|r_1|} \frac{\frac{((\beta-\gamma)^2) - (\omega^2)}{2} \widehat{\omega} - (\beta - \gamma, \omega)(\widehat{\beta} - \widehat{\gamma})}{\left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2}, \\ \widetilde{\beta} &:= \gamma' - \frac{1}{|r_1|} \frac{\frac{((\beta-\gamma)^2) - (\omega^2)}{2} (\widehat{\beta} - \widehat{\gamma}) - (\beta - \gamma, \omega) \widehat{\omega}}{\left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2}. \end{aligned}$$

Then $(\widetilde{\beta}, \widetilde{\omega}) \in \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$.

Proof. By our assumption, $\widehat{\omega} \in P^+(X)_\mathbb{R}$. It is sufficient to prove $(\widetilde{\omega}, \widetilde{\omega}) > 0$ and $(\widetilde{\omega}, \widehat{\omega}) > 0$, which follows from the following equations:

$$(3.12) \quad \begin{aligned} (\widetilde{\omega}^2) &= \frac{1}{|r_1|^2} \frac{(\omega^2)}{\left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2}, \\ (\widetilde{\omega}, \widehat{\omega}) &= \frac{1}{|r_1|} \frac{(\omega^2)^2 + (\beta - \gamma, \omega)^2 - (D^2)(\omega^2)}{2 \left(\left(\frac{((\beta-\gamma)^2) - (\omega^2)}{2} \right)^2 + (\beta - \gamma, \omega)^2 \right)}, \end{aligned}$$

where $\beta - \gamma = \lambda\omega + D$ ($\lambda \in \mathbb{R}$, $D \in \omega^\perp$). \square

By [19, sect. A.1], we get the following commutative diagram:

$$(3.13) \quad \begin{array}{ccc} H^*(X, \mathbb{Q})_{\text{alg}} & \longrightarrow & H^*(X, \mathbb{Q})_{\text{alg}} \\ Z_{(\beta, \omega)} \downarrow & & \downarrow Z_{(\widetilde{\beta}, \widetilde{\omega})} \\ \mathbb{C} & \xrightarrow{\zeta^{-1}} & \mathbb{C} \end{array}$$

where

$$\zeta = r_1 \left(\frac{((\gamma - \beta)^2) - (\omega^2)}{2} + \sqrt{-1}(\beta - \gamma, \omega) \right).$$

Proposition 3.9. *For $(\beta, \omega) \in \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$, we have $\Phi(\xi(\beta, \omega)) = \xi(\widetilde{\beta}, \widetilde{\omega})$.*

Proof. The proof is completely the same as of [29, Prop. 3.7]. \square

4. STABILITY CONDITIONS ON A BLOW-UP

4.1. Stability conditions for (β, tH) . Let $\pi : X \rightarrow Y$ be the blow-up of a point as in section 2. In this subsection, we shall study the map ξ in a neighborhood of $H + (H, \delta)\varrho_X$, where $H \in \pi^*(\text{Amp}(Y)_{\mathbb{Q}})$. We start with the following easy fact.

Lemma 4.1. *Assume that $\omega \in \pi^*(\text{Amp}(Y)_{\mathbb{R}})$.*

- (1) $Z_{(\beta, \omega)}(\mathcal{O}_C(-a)) = 0$ if and only if $(\beta, C) = -a + \frac{1}{2}$.
- (2) If $(\beta - \delta, \omega) \neq 0$, then $Z_{(\beta, \omega)}(\mathcal{O}_C(-a)) = 0$ if and only if $\xi(\beta, \omega) \in v(\mathcal{O}_C(-a))^{\perp}$.

Proof. (1) is obvious. Since $Z_{(\beta, \omega)}(v) \notin \mathbb{R}$ for $(\beta - \delta, \omega) \neq 0$, $\xi(\beta, \omega) \in v(\mathcal{O}_C(-a))^{\perp}$ if and only if $Z_{(\beta, \omega)}(\mathcal{O}_C(-a)) = 0$. Thus (2) holds. \square

Lemma 4.2. (1) *For $u = H + (H, \delta)\varrho_X$ with $H \in \pi^*(\text{Amp}(Y)_{\mathbb{Q}})$,*

$$(4.1) \quad \xi^{-1}(u) \cap \overline{\mathcal{K}}(X) = \{(\beta, \omega) \mid \beta - \delta \in H^{\perp}, (\beta, C) \notin \frac{1}{2} + \mathbb{Z}, \omega \in \mathbb{R}_{>0}H\}.$$

In particular, $\xi^{-1}(u) \cap \overline{\mathcal{K}}(X)$ is not connected.

- (2) *Assume that $\text{rk } u = -1$.*

(a) *If $u \notin \cup_{a \in \mathbb{Z}} v(\mathcal{O}_C(-a))^{\perp}$, then*

$$(4.2) \quad \xi^{-1}(u) \cap \overline{\mathcal{K}}(X) = \{(\beta, \omega) \in \xi^{-1}(u) \mid \omega \in \text{Amp}(X)_{\mathbb{R}} \cup \pi^*(\text{Amp}(Y)_{\mathbb{R}})\}.$$

(b) *If $u \in \cup_{a \in \mathbb{Z}} v(\mathcal{O}_C(-a))^{\perp}$, then*

$$(4.3) \quad \xi^{-1}(u) \cap \overline{\mathcal{K}}(X) = \{(\beta, \omega) \in \xi^{-1}(u) \mid \omega \in \text{Amp}(X)_{\mathbb{R}}\}.$$

In particular, $\xi^{-1}(u) \cap \overline{\mathcal{K}}(X)$ is connected.

We fix $H \in \pi^*(\text{Amp}(Y)_{\mathbb{Q}})$. We have an inclusion

$$\iota : H^{\perp} \times \mathbb{R} \rightarrow \text{NS}(X)_{\mathbb{R}} \times P^+(X)_{\mathbb{R}}$$

such that $\iota(D, s) = (\delta + sH + D, H)$. By (3.6),

$$(4.4) \quad \xi(\delta + sH + D, H) = e^{\delta} \left\{ \frac{r^2((H^2)(s^2 + 1) - (D^2)) + (v^2)}{2r^2} H + s(H^2) \left(D + \left(1 + \frac{(v^2)}{2r^2} \varrho_X \right) \right) \right\}.$$

$\xi(\text{im } \iota)$ consists of $e^{\delta}(H + X + y(1 + \frac{(v^2)}{2r^2})\varrho_X)$ satisfying

$$(4.5) \quad (H^2) \geq -(X^2) + y^2((H^2) + \frac{(v^2)}{r^2}), y \neq 0$$

or $(X, y) = (0, 0)$. Moreover if $s < 0$, then $y < 0$. For a fixed D , $\xi \circ \iota(D, s) = \xi(\delta + sH + D, H)$ is a line passing $H + (H, \delta)\varrho_X$ and

$$(4.6) \quad \iota^{-1}(\xi^{-1}(H + (H, \delta)\varrho_X)) = \{(D, 0) \mid D \in H^{\perp}\}.$$

We also have

$$\text{im } \iota \cap \overline{\mathcal{K}}(X) = \{(\delta + sH + D, H) \mid (\delta + D, C) \notin \frac{1}{2} + \mathbb{Z}\}.$$

Lemma 4.3. *Assume that $(\beta, \omega), (\beta', \omega') \in \overline{\mathcal{K}}(X)$ satisfy $\xi(\beta, \omega) = \xi(\beta', \omega') \notin \cup_{a \in \mathbb{Z}} v(\mathcal{O}_C(-a))^{\perp}$. Then $\mathcal{M}_{(\beta, \omega)}(v) = \mathcal{M}_{(\beta', \omega')}(v)$.*

Proof. By Lemma 4.2 (2) and Lemma 3.3, the claim follows. \square

Remark 4.4. Assume that $H \in \pi^*(\text{Amp}(Y))$. Then the category $\mathfrak{C}^{\beta'}$ is determined by the integer l satisfying $l - \frac{1}{2} < (\beta', C) < l + \frac{1}{2}$, where

$$(4.7) \quad (\beta', C) = \frac{r^2((\omega^2) - ((\beta - \delta)^2)) + (v^2)}{2r^2(\beta - \delta, \omega)}(\omega, C) + (\beta, C).$$

Indeed by (3.2) and the proof of Lemma 3.8,

$$\beta' \equiv \beta + \frac{r^2((\omega^2) - ((\beta - \delta)^2)) + (v^2)}{2r^2(\beta - \delta, \omega)}\omega \pmod{\mathbb{R}H}.$$

Remark 4.5. For the fiber $\xi^{-1}(H + (H, \delta)\varrho_X)$, Lemma 4.3 does not hold by Lemma 4.1 (1). Moreover if $(\omega, C) < 0$, then Lemma 4.3 does not hold. Thus the structure of $\text{Stab}(X)$ seems to be complicated if $(\omega, C) < 0$.

By Proposition 6.11, we get the following claim.

Proposition 4.6. Assume that (β, tH) belongs to an adjacent chamber of ϱ_X^\perp and $(\beta - \delta, H) < 0$. Then $\mathcal{M}_{(\beta, tH)}(v) = \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v)$.

Proof. We can take a sufficiently large t' such that $\xi(\beta, t'H)$ belongs to the adjacent chamber \mathcal{C} . Since (β, tH) also belongs to \mathcal{C} , we have $\mathcal{M}_{(\beta, tH)}(v) \cong \mathcal{M}_{(\beta, t'H)}(v)$. By Proposition 6.11, $\mathcal{M}_{(\beta, t'H)}(v) \cong \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v)^{ss}$. Thus the claim holds. \square

The following claim gives an explicit example of (β, tH) in Proposition 4.6

Proposition 4.7. We set $\mu := \min\{(D, H) > 0 \mid D \in \text{NS}(X)\}$. Assume that v satisfies $\gcd(r, (r\delta, H)/\mu) = 1$. We take $r_0 \in \mathbb{Z}$ and $\xi_0 \in \pi^*(\text{NS}(Y))$ such that $(r\xi_0 - r_0(r\delta), H) = -\mu$. Then

$$\mathcal{M}_{(\delta + sH + D, tH)}(v) = \mathcal{M}_H^{\delta + D - \frac{1}{2}K_X}(v)$$

if

$$(4.8) \quad -\frac{\mu}{rr_0(H^2)} \leq s < 0.$$

Proof. Assume that s satisfies (4.8). We set

$$(4.9) \quad \begin{aligned} \gamma &:= \delta + D + \frac{(\xi_0 - r_0\delta, H)}{r_0(H^2)}H \\ &= \delta + D - \frac{\mu}{rr_0(H^2)}H. \end{aligned}$$

Then $r_0(\gamma, H) = (\xi_0, H)$ and $(c_1(E), H) - \text{rk } E(\gamma, H) \in \frac{\mu}{r_0}\mathbb{Z}$ for all $E \in \mathbf{D}(X)$. Since $d_\gamma(v)(H^2) = \frac{\mu}{r_0}$, there is no wall intersecting $\{(sH + D, tH) \mid t > 0\}$. Hence the claim holds. \square

Toda [23], [24] constructed a stability condition $\sigma_{(0, \pi^*(\omega') - qC)} = (Z_{(0, \pi^*(\omega') - qC)}, \mathcal{A}_{\omega'}(X/Y))$, where $q < 0$, $Z_{\pi^*(\omega') - qC}(E) = (e^{\pi^*(\omega') - qC}, v(E))$, $\mathcal{A}_{\omega'}(X/Y) = \langle \mathbf{L}\pi^*\mathcal{A}_{(0, \omega')}, \mathcal{C}_{X/Y}^0 \rangle$ and $\mathcal{C}_{X/Y}^0$ is spanned by $\mathcal{O}_C[-1]$. By the same proof in [24], we shall generalize $\mathcal{A}_{\omega'}(X/Y)$ as $\mathcal{A}_{(\beta', \omega')}(X/Y) := \langle \mathbf{L}\pi^*\mathcal{A}_{(\beta', \omega')}, \mathcal{O}_C[-1] \rangle$, where $(\beta', \omega') \in \text{NS}(Y)_\mathbb{Q} \times \text{NS}(Y)_\mathbb{Q}$. Then

$$\sigma_{(\pi^*(\beta'), \pi^*(\omega') - qC)} := (Z_{(\pi^*(\beta'), \pi^*(\omega') - qC)}, \mathcal{A}_{(\beta', \omega')}(X/Y))$$

is a stability condition. Assume that $v = \pi^*(v')$ with $v' \in H^*(Y, \mathbb{Q})$. Then $\mathcal{M}_{(\pi^*(\beta'), H - qC)}(v) \cong \mathcal{M}_{(\beta', H')}(v')$, where $H = \pi^*(H')$.

4.2. A classification of walls. For $(s, q) \in \mathbb{R}^2$ with $|s|, |q| \ll 1$, we consider

$$(4.10) \quad \begin{aligned} \xi(\delta + sH + pC, H - qC) &= e^\delta \left(\frac{(1 + s^2)(H^2) + p^2 - q^2 + 2psq + \frac{(v^2)}{r^2}}{2} H \right. \\ &\quad \left. + \frac{q^3 + q(s^2 - 1)(H^2) + p^2q + 2ps(H^2) - \frac{(v^2)}{r^2}q}{2} C + (s(H^2) + pq)(1 + \frac{(v^2)}{2r^2}\varrho_X) \right). \end{aligned}$$

If $(s, q) = (0, 0)$, then $\xi(\delta + sH + pC, H - qC) = e^\delta H \in C^+(v)$. In this subsection, we shall classify walls in a neighborhood of $(s, q) = (0, 0)$. We set

$$(4.11) \quad L := \{(\delta + sH + pC, H - qC) \mid (s, q) \in \mathbb{R}^2\}.$$

We set $\epsilon := s(H^2) + pq$ and assume that $\epsilon \leq 0$. We have

$$(4.12) \quad \xi(\delta + sH + pC, H - qC) = e^\delta (H + xC + y(1 + \frac{(v^2)}{2r^2}\varrho_X)),$$

where

$$(4.13) \quad \begin{aligned} x &:= \frac{q^3 + q(s^2 - 1)(H^2) + p^2q + 2ps(H^2) - \frac{(v^2)}{r^2}q}{(1 + s^2)(H^2) + p^2 - q^2 + 2psq + \frac{(v^2)}{r^2}}, \\ y &:= \frac{2\epsilon}{(1 + s^2)(H^2) + p^2 - q^2 + 2psq + \frac{(v^2)}{r^2}}. \end{aligned}$$

Then we have expansions

$$(4.14) \quad \begin{aligned} x &= \frac{-((H^2) + p^2 + \frac{(v^2)}{r^2})q + 2p\epsilon}{(H^2) + p^2 + \frac{(v^2)}{r^2}} + O_2(\epsilon, q), \\ y &= \epsilon \left(\frac{2}{(H^2) + p^2 + \frac{(v^2)}{r^2}} + O_1(\epsilon, q) \right), \end{aligned}$$

where $O_n(\epsilon, q)$ is a power series of ϵ and q contained in the ideal $(\epsilon, q)^n$. If $q = 0$, then

$$(4.15) \quad \begin{aligned} x &= \frac{2ps(H^2)}{(1+s^2)(H^2) + p^2 + \frac{(v^2)}{r^2}} = \frac{2s(H^2)}{p} + O_2(p^{-1}), \\ y &= \frac{2\epsilon}{(1+s^2)(H^2) + p^2 + \frac{(v^2)}{r^2}} = \frac{2\epsilon}{p^2} + O_3(p^{-1}), \\ x &= py. \end{aligned}$$

If $p \gg 0$, then (x, y) is close to $y = 0$.

We set $\beta_0 := \delta + p_0C$, and assume that $-\frac{1}{2} + l < (\beta_0, C) < \frac{1}{2} + l$ ($l \in \mathbb{Z}$). By (4.14), we can take a neighborhood U of (β_0, H) in L such that \overline{U} is compact and $\xi : U \rightarrow \xi(U)$ is isomorphic, where $p := p_0$. In particular $\xi(U)$ is a neighborhood of $\xi(\beta_0, H) = e^\delta H$. By shrinking U , we may assume that there are finitely many Mukai vectors defining walls in U and all walls passes (β_0, H) . For each Mukai vector v_1 defining a wall, we may also assume that all walls in U with respect to v_1 passes (β_0, H) . We set

$$(4.16) \quad \begin{aligned} U^{\leq 0} &:= U \cap \{(\delta + sH + pC, H - qC) \mid \epsilon \leq 0\}, \\ U^{< 0} &:= U \cap \{(\delta + sH + pC, H - qC) \mid \epsilon < 0\}. \end{aligned}$$

Let E_1 be a $\sigma_{(\beta, \omega)}$ -stable object defining a wall in $U^{\leq 0}$. Since there is no wall with respect to $v(E_1)$ between (β, ω) and (β_0, H) , E_1 is $\sigma_{(\beta_0, H)}$ -semi-stable. Since $(\beta_0 - \delta, H) = 0$, E_1 is generated by μ -semi-stable objects and objects $\mathcal{O}_C(l)[-1], \mathcal{O}_C(l-1), k_x \in \mathfrak{C}^l[-1]$, ($x \in X \setminus C$).

Since $\varrho_X^\perp \cap e^\delta(\mathbb{R}H + \mathbb{R}C + \mathbb{R}(1 + \frac{(v^2)}{2r^2}\varrho_X))$ is $y = 0$, if $|\epsilon| \ll q$, then $\sigma_{(\beta, \omega)}$ -twisted stability coincides with Gieseker semi-stability with respect to $H - qC$, where $(\beta, \omega) = (\delta + sH + p_0C, H - qC)$.

Lemma 4.8. *If $\text{rk } E_1 \neq 0$, then E_1 is a μ -semi-stable object of \mathfrak{C}^l .*

Proof. For $(\beta, \omega) \in U^{< 0}$, E_1 is $\sigma_{(\beta, \omega)}$ -stable with $\phi_{(\beta, \omega)}(E_1) \in (0, 1]$. Hence $H^i(E_1) = 0$ for $i \neq -1, 0$. Since E_1 is $\sigma_{(\beta_0, H)}$ -semi-stable with $\phi_{(\beta_0, H)}(E_1) = 0$, $H^i(E_1) = 0$ for $i \neq 0, 1$. Therefore $E_1 \in \mathfrak{C}^l$. Thus E_1 is a μ -semi-stable object of \mathfrak{C}^l . \square

All walls in $\xi^{-1}(\xi(U)) \cap \mathcal{K}(X)$ are defined by an object E_1 which defines a wall in U .

4.2.1. *Stability of $\mathcal{O}_C(a)$.* For $p_1 > p_0 = (\delta - \beta_0, C)$, by shrinking U , we may assume that

$$(4.17) \quad \{\xi(\beta_0 + sH, H - qC) \mid \epsilon \leq 0, q > 0\} \supset \{e^\delta(H + xC + y(1 + \frac{(v^2)}{2r^2}\varrho_X)) \in \xi(U) \mid x < p_1y, y \leq 0\},$$

where $\epsilon = s(H^2) + p_0q$ and we also use

$$x = -q + p_0y + O_2(\epsilon, q).$$

For $a \in \mathbb{Z}$ with $p_1 < a + (\delta, C) - \frac{1}{2}$, we take $p := p(a) \in \mathbb{R}$ such that $a + (\delta, C) - \frac{3}{2} < p(a) < a + (\delta, C) - \frac{1}{2}$. We set $\beta_p := \delta + p(a)C$. Let V_a be the open set defined by $x < (a + (\delta, C) - \frac{3}{2})y$ and $y < 0$. Since $(-a + 1) - \frac{1}{2} < (\beta_p, C) < (-a + 1) + \frac{1}{2}$ and $\mathcal{O}_C(-a)[1]$ is an irreducible object of \mathfrak{C}^{-a+1} , $\mathcal{O}_C(-a)[1]$ is $\sigma_{(\beta_p, H)}$ -stable. Hence there is an open neighborhood U_p of (β_p, H) such that $\mathcal{O}_C(-a)$ is a $\sigma_{(\beta_p + sH, H - qC)}$ -stable object for $(\beta_p + sH, H - qC) \in U_p$. By shrinking U_p , we may assume that

$$\ell_a := \{e^\delta(H + xC + y(1 + \frac{(v^2)}{2r^2}\varrho_X)) \in \xi(U_p) \mid x = (a + (\delta, C) - \frac{1}{2})y, y < 0\}$$

is a subset of

$$\{\xi(\beta_p + sH, H - qC) \mid 0 > \epsilon_p := s(H^2) + p(a)q, q > 0\}.$$

We note that $v(\mathcal{O}_C(-a))^\perp$ is the line $x = (a + (\delta, C) - \frac{1}{2})y$. By Remark 3.6, $\xi_v^{-1}(\ell_a) = \xi_{v(\mathcal{O}_C(-a))}^{-1}(\ell_a)$. Applying Lemma 3.3 to $\xi_{v(\mathcal{O}_C(-a))}^{-1}(\ell_a)$, $\mathcal{O}_C(-a)$ is $\sigma_{(\beta_0 + sH, H - qC)}$ -stable, if $(\beta_0 + sH, H - qC) \in U$ ($q > 0$) belongs to $\xi^{-1}(\ell_a)$. Thus we obtain the following.

Lemma 4.9. *For any a with $p_1 < a + (\delta, C) - \frac{1}{2}$, there is a neighborhood U_a of $\xi^{-1}(\ell_a) \cap U$ such that $\mathcal{O}_C(-a)$ is $\sigma_{(\beta_0 + sH, H - qC)}$ -stable. Moreover*

$$\phi_{(\beta_0 + sH, H - qC)}(\mathcal{O}_C(-a)) = \phi_{(\beta_0 + sH, H - qC)}(E), \quad E \in \mathcal{M}_{(\beta_0 + sH, H - qC)}(v)$$

if $(\beta_0 + sH, H - qC) \in \xi^{-1}(\ell_a) \cap U$.

4.3. A description of $\mathcal{M}_{(\delta+sH+pC, H-qC)}(v)$. For $(\beta, \omega) \in U^{<0}$ with $q \geq 0$, we shall describe $\mathcal{M}_{(\beta, \omega)}(v)$ in terms of the moduli spaces of semi-stable perverse coherent sheaves $\mathcal{M}_H^\gamma(v)$. We set

$$u := e^\delta(H + xC + y(1 + \frac{(v^2)}{2r^2}\varrho_X)).$$

We first assume that $u \in \xi(U)$ satisfies $(u, v(\mathcal{O}_C(-a))) = -x - y(\frac{1}{2} - a - (\delta, C)) \neq 0$ for all a . Under this condition, we shall describe $\mathcal{M}_{(\beta, \omega)}(v)$ with $\xi(\beta, \omega) = u$, where $(\beta, \omega) = (\delta + sH + pC, H - qC)$. There are $s' < 0, t' > 0$ such that $\xi(\delta + s'H + p'C, t'H) = u$, where

$$(4.18) \quad p' := \frac{x}{y} = \frac{q^3 + q(s^2 - 1)(H^2) + p^2q + 2ps(H^2) - \frac{(v^2)}{r^2}q}{2\epsilon}.$$

Then $\{\xi(\delta + sH + p'C, tH) \mid s \leq 0, t > 0\}$ is a segment. Since there is no wall between u and $e^\delta H$,

$$\mathcal{M}_{(\beta, \omega)}(v) = \mathcal{M}_{(\delta+s'H+p'C, t_\infty H)}(v) = \mathcal{M}_H^{\delta+p'C-\frac{1}{2}K_X}(v),$$

where t_∞ is sufficiently large. If (β, ω) is not general, then the wall is defined by

$$(4.19) \quad \frac{\chi(E(-(\delta + \frac{x}{y}C - \frac{1}{2}K_X)))}{\text{rk } E} = \frac{\chi(E_1(-(\delta + \frac{x}{y}C - \frac{1}{2}K_X)))}{\text{rk } E_1},$$

where E_1 is a $(\delta + p'C - \frac{1}{2}K_X)$ -twisted stable object with $(c_1(E_1) - \text{rk } E_1 \delta, H) = 0$.

We next consider the case where $(\beta, C) = l - \frac{1}{2}$. We set $\beta_\pm := \beta \mp \eta C$ for a sufficiently small $\eta > 0$.

Definition 4.10. For $E \in \mathcal{A}_{(\beta_-, tH)}$ with $Z_{(\beta, tH)}(E) \in \mathbb{H}$, E is $\sigma_{(\beta, tH)}$ -semi-stable, if

$$Z_{(\beta, tH)}(E_1)/Z_{(\beta, tH)}(E) \in -\mathbb{H} \cup \mathbb{R}_{\geq 0}$$

for all non-zero subobject E_1 of E in $\mathcal{A}_{(\beta_-, tH)}$. Let $\mathcal{M}_{(\beta, tH)}(v)$ be the moduli stack of $\sigma_{(\beta, tH)}$ -semi-stable objects E with $v(E) = v$.

If $v = nv(\mathcal{O}_C(l-1))$, then we also set $\mathcal{M}_{(\beta, tH)}(v) := \mathcal{M}_{(\beta_-, tH)}(v) = \{\mathcal{O}_C(l-1)^{\oplus n}\}$. Then we have $\mathcal{M}_{(\beta_+, tH)}(v) = \mathcal{M}_{(\beta, tH)}(v)$.

Proposition 4.11. $\mathcal{M}_{(\gamma, \omega)}(v) = \mathcal{M}_{(\beta, tH)}(v) = \mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v)$ for $(\gamma, \omega) \in \xi^{-1}(\xi(\beta, tH))$.

Proof. For $E \in \mathcal{M}_{(\beta, tH)}(v)$, we take a filtration

$$(4.20) \quad 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$$

such that $E_i := F_i/F_{i-1}$ are $\sigma_{(\beta, tH)}$ -stable and $Z_{(\beta, tH)}(E_i) = \lambda_i Z_{(\beta, tH)}(E)$ with $0 \leq \lambda_i \leq 1$. If $\lambda_i = 0$, then $E_i = \mathcal{O}_C(l-1)$. We set $v_i := v(E_i)$. For the proof of our claim, it is sufficient to prove that

$$(4.21) \quad \mathcal{M}_{(\gamma, \omega)}(v_i)^s = \mathcal{M}_{(\beta, tH)}(v_i)^s = \mathcal{M}_H^{\beta-\frac{1}{2}K_X}(v_i)^s$$

for $(\gamma, \omega) \in \xi^{-1}(\xi(\beta, tH))$.

We first assume that $\lambda_i \neq 0$. As in Remark 3.6, we set $\xi_{v_i}(\gamma, \omega) := \text{Im}(Z_{(\gamma, \omega)}(v_i)^{-1}e^{\gamma+\sqrt{-1}\omega})$. Since $\text{Im}(Z_{(\beta, tH)}(v_i)^{-1}e^{\beta+\sqrt{-1}tH}) = \lambda_i^{-1}\text{Im}(Z_{(\beta, tH)}(v)^{-1}e^{\beta+\sqrt{-1}tH})$, we get $\xi_{v_i}(\beta, tH) \in \mathbb{R}\xi(\beta, tH)$. By Remark 3.6, $\xi_{v_i}^{-1}(\xi_{v_i}(\beta, tH)) = \xi^{-1}(\xi(\beta, tH))$. We take $(\gamma_\pm, \omega_\pm) \in \xi_{v_i}^{-1}(\xi_{v_i}(\beta_\pm, tH))$ which are in a neighborhood of (γ, ω) . Then $\mathcal{M}_{(\beta_\pm, tH)}(v_i)^s = \mathcal{M}_{(\gamma_\pm, \omega_\pm)}(v_i)^s$. Since

$$\mathcal{M}_{(\gamma_-, \omega_-)}(v_i)^s \cap \mathcal{M}_{(\gamma_+, \omega_+)}(v_i)^s = \mathcal{M}_{(\gamma, \omega)}(v_i)^s,$$

E_i is $\sigma_{(\gamma, \omega)}$ -stable with $Z_{(\gamma, \omega)}(E_i) \in \mathbb{R}_{>0}Z_{(\gamma, \omega)}(v)$. If $E_i = \mathcal{O}_C(l-1)$, then it is also $\sigma_{(\gamma, \omega)}$ -stable with $Z_{(\gamma, \omega)}(E_i) \in \mathbb{R}_{>0}Z_{(\gamma, \omega)}(v)$. Indeed we also have $\mathcal{M}_{(\gamma_\pm, \omega_\pm)}(v_i) = \mathcal{M}_{(\beta_\pm, tH)}(v_i) = \{\mathcal{O}_C(l-1)\}$ and $\mathcal{M}_{(\gamma, \omega)}(v_i) = \mathcal{M}_{(\beta, tH)}(v_i) = \{\mathcal{O}_C(l-1)\}$. Hence E is $\sigma_{(\gamma, \omega)}$ -semi-stable with a Jordan-Hölder filtration (4.20). Since ω is ample, by our choice of U , $e^\beta H \in v_i^\perp$ and all walls for v_i passes $e^\delta H$. Hence

$$v_i = e^\beta(r_i + d_i H + D_i + a_i \varrho_X), \quad r_i = r \frac{d_i}{d}, \quad a_i = a \frac{d_i}{d}$$

and

$$\mathcal{M}_H^{\beta_\pm - \frac{1}{2}K_X}(v_i)^s = \mathcal{M}_{(\beta_\pm, tH)}(v_i)^s = \mathcal{M}_{(\gamma_\pm, \omega_\pm)}(v_i)^s.$$

By Lemma 2.15, we have

$$\mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v_i)^s = \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v_i)^s \cap \mathcal{M}_H^{\beta_+ - \frac{1}{2}K_X}(v_i)^s.$$

Hence (4.21) holds. □

5.1. Ample line bundles on the moduli spaces. Assume that there is a moduli scheme $M_{(\beta,\omega)}(v)$ of S -equivalence classes of semi-stable objects and consisting of stable objects. For simplicity, we assume that there is a universal family \mathcal{E}_v on $M_{(\beta,\omega)}(v) \times X$. For $\alpha \in K(X)_{\mathbb{Q}}$, we set

$$(5.1) \quad \mathcal{L}(\alpha) := \det p_{M_{(\beta,\omega)}(v)!}(\alpha^\vee \otimes \mathcal{E}_v).$$

We set

$$(5.2) \quad K(X)_v := \{\alpha \in K(X)_{\mathbb{Q}} \mid \chi(\alpha^\vee \otimes v) = 0\}.$$

We have a morphism

$$(5.3) \quad \begin{array}{ccc} \kappa : & K(X)_v & \rightarrow \text{Pic}(M_{(\beta,\omega)}(v))_{\mathbb{Q}} \\ & \alpha & \mapsto \mathcal{L}(\alpha). \end{array}$$

As in [27], we have a morphism $\mathbf{a} : M_{(\beta,\omega)}(v) \rightarrow \text{Alb}(X) \times \text{Pic}^0(X)$. κ induces a homomorphism

$$(5.4) \quad \theta_v : v^\perp \rightarrow \text{NS}(M_{(\beta,\omega)}(v))_{\mathbb{Q}} / \mathbf{a}^*(\text{NS}(\text{Alb}(X) \times \text{Pic}^0(X))_{\mathbb{Q}}),$$

where

$$(5.5) \quad \begin{aligned} v^\perp &:= \{u \in H^*(X, \mathbb{Q})_{\text{alg}} \mid \chi(u(\text{td}_X^{-1})^\vee, v) = 0\} \\ &= \{u \in H^*(X, \mathbb{Q})_{\text{alg}} \mid (u, v) = 0\}. \end{aligned}$$

By works of Fogarty [8] and Li [12], [14] (see also [27]), if $\text{rk } v = 1, 2$ and $M_{(\beta,\omega)}(v)$ is the moduli of Gieseker semi-stable sheaves $M_{\omega}^{\beta - \frac{1}{2}K_X}(v)$, then \mathbf{a} is the Albanese morphism and θ_v is isomorphic for all sufficiently large (v^2) depending on $\omega, \text{rk } v, c_1(v)$.

Remark 5.1. If there is a universal family as a twisted object, then κ is well-defined.

We take $\xi_1, \xi_2 \in K(X)_{\mathbb{Q}}$ such that

$$(5.6) \quad \begin{aligned} \text{ch}(\xi_1) &= (H + (H, \delta)\varrho_X)(\text{td}_X^{-1})^\vee = H + (H, \delta - \tfrac{1}{2}K_X)\varrho_X, \\ \text{ch}(\xi_2) &= - \left(e^{\beta - \frac{1}{2}K_X} - \frac{\chi(e^{\beta - \frac{1}{2}K_X}, v)}{r} \varrho_X \right). \end{aligned}$$

By the construction of the moduli scheme, we have the following.

Lemma 5.2 (cf. [19, Lem. A.4]). $\mathcal{L}(n\xi_1 + \xi_2)$ is ample for $n \gg 0$.

We set

$$(5.7) \quad u := e^\delta(\zeta - (1 + \frac{v^2}{2r^2})\varrho_X).$$

Assume that there is no wall between $H + (H, \delta)\varrho_X$ and u . Then there is a semi-circle in (s, t) -plane such that $\xi(\beta, tH) = u$, where $\beta = \delta + sH - D$, $D = \zeta - \frac{(\zeta, H)}{(H^2)}H$. Then Lemma 3.8 implies

$$M_{(\delta+sH-D, tH)}(v) = M_H^{\beta - \frac{1}{2}K_X}(v).$$

Since $\beta = \delta - \zeta + \left(s - \frac{(\zeta, H)}{(H^2)}\right)H$, we also have

$$M_{(\delta+sH-D, tH)}(v) = M_H^{\delta - \zeta - \frac{1}{2}K_X}(v).$$

Proposition 5.3. Assume that there is no wall between $H + (H, \delta)\varrho_X$ and $u := \zeta + (\zeta, \delta)\varrho_X - (e^\delta + \frac{v^2}{2r^2})\varrho_X$. Then for $\alpha \in K(X)_{\mathbb{Q}}$ with $\text{ch}(\alpha) = u(\text{td}_X^{-1})^\vee$, $\mathcal{L}(\alpha)$ is a \mathbb{Q} -ample divisor. In particular, for an adjacent chamber \mathcal{C} of ϱ_X^\perp , $\theta_v(\mathcal{C}) \subset \text{Amp}(M_H^\gamma(v))_{\mathbb{Q}}$, where $\gamma = \delta - \zeta - \frac{1}{2}K_X$.

Proof. Let (β, tH) satisfy $\xi(\beta, tH) = u$ as above, where $\beta \equiv \delta - \zeta \pmod{\mathbb{R}H}$. Then we see that

$$(5.8) \quad \begin{aligned} u &= -e^\delta(1 - \zeta + \frac{v^2}{2r^2})\varrho_X \\ &\equiv -(1 + \beta + ((\beta, \delta) - \frac{a}{r})\varrho_X) \pmod{\mathbb{R}(H + (H, \delta)\varrho_X)} \\ &= - \left(e^\beta + \frac{(e^\beta, v)}{r} \varrho_X \right) \pmod{\mathbb{R}(H + (H, \delta)\varrho_X)} \end{aligned}$$

by (1.9), where $v = r(1 + \delta + \frac{a}{r}\varrho_X)$. Hence

$$(5.9) \quad \begin{aligned} u(\text{td}_X^{-1})^\vee &\equiv - \left(e^{\beta - \frac{1}{2}K_X} - \frac{\chi(e^{\beta - \frac{1}{2}K_X}, v)}{r} \varrho_X \right) \pmod{\mathbb{R} \text{ch}(\xi_1)} \\ &\equiv \text{ch}(\xi_2) \pmod{\mathbb{R} \text{ch}(\xi_1)}. \end{aligned}$$

We take u' such that $u' = u - \epsilon(H + (H, \delta)\varrho_X)$ ($\epsilon > 0$) belongs to the same chamber. We take α' with $\text{ch}(\alpha') = u'(\text{td}_X^{-1})^\vee$. Then

$$c_1(\mathcal{L}(\alpha')) = c_1(p_{M_H^\gamma(v)}^*(u'^\vee \text{ch } \mathcal{E}_v)).$$

By [3], $\mathcal{L}(\alpha')$ is nef. We take an ample \mathbb{Q} -divisor $\mathcal{L}(\xi_1 + \frac{1}{n}\xi_2)$, $n \gg 0$. Since $\mathcal{L}(\alpha) = \mathcal{L}((1-t)\alpha' + t(\xi_1 + \frac{1}{n}\xi_2))$ ($1 > t > 0$), Lemma 5.2 implies $\mathcal{L}(\alpha)$ is ample. \square

Remark 5.4. If $\xi(\beta, \omega)$ belongs to a wall v_1^\perp of \mathcal{C} and there are $\sigma_{(\beta, \omega)}$ -stable objects E_1 and E_2 with $v(E_1) = v_1$ and $v(E_2) = v - v_1$ such that $\dim \text{Ext}^1(E_2, E_1) \geq 2$ and $\phi_{(\beta', \omega')}(E_1) < \phi_{(\beta', \omega')}(E_2)$ for $\xi(\beta', \omega') \in \mathcal{C}$. Then $\theta_v(\xi(\beta, \omega))$ is not ample.

Let $N_H(v)$ be the Uhlenbeck moduli space of μ -semi-stable sheaves constructed by Li [13] (see also Remark 5.6). Thus a point of $N_H(v)$ corresponds to a pair (F, Z) of a poly-stable vector bundle F with respect to H and a 0-cycle $Z = \sum_{i=1}^p n_i x_i \in S^n X$ such that $v = v(F) - n\varrho_X$, where $n \geq 0$. For $(F, \sum_{i=1}^p n_i x_i) \in N_H(v)$, we have $F \oplus (\bigoplus_{i=1}^p k_{x_i}[-1]^{\oplus n_i}) \in \mathcal{M}_{(\delta, H)}(v)$. Hence $N_H(v)$ parameterizes S -equivalence classes of $\sigma_{(\delta, \omega)}$ -semi-stable objects. By the construction of the moduli space, we have a morphism $f_H : M_H^{\beta - \frac{1}{2}K_X}(v) \rightarrow N_H(v)$ and $\mathcal{L}(\alpha) \in f_H^*(\text{Pic}(N_H(v)))$ if $\text{ch}(\alpha) = (H + (H, \delta)\varrho_X)(\text{td}_X^{-1})^\vee$.

Remark 5.5. By the construction of $N_H(v)$ in [13], $N_H(v)$ may depend on the choice of H . Assume that $\mathcal{M}_H(v)^{\mu\text{-ss}}$ is irreducible, normal and $\mathcal{M}_H(v)^{\mu\text{-s}}$ is an open dense substack. Then we can construct $N_H(v)$ as a normal projective variety and $f_H : M_H^\gamma(v) \rightarrow N_H(v)$ is a birational map. If H' and H belongs to the same chamber, then we have a birational map $f_{H'} \circ f_H^{-1} : N_H(v) \rightarrow N_{H'}(v)$. Since $f_{H'} \circ f_H^{-1}(z)$ is a point for all $z \in N_H(v)$, it is an isomorphism. In particular, if $(v^2) \gg 0$, then $N_H(v)$ depends only on the chamber. By [10, 1.4], $\mathcal{L}(\alpha) \in f_H^*(\text{Pic}(N_H(v)))$ if $\text{rk } \alpha = 0$.

By the same argument, we have a contraction $N_H(v) \rightarrow N_{H'}(v)$ if H belongs to a chamber and H' is on the boundary of the chamber.

Remark 5.6. We have a morphism $f : M_H^\gamma(v) \rightarrow \text{Pic}^0(X)$ by sending E to $\det(E - E_0)$, where E_0 is a fixed element of $M_H^\gamma(v)$. In the construction of the Uhlenbeck moduli space [13], the determinant $\det E$ is fixed. In order to construct the Uhlenbeck moduli space for $M_H^\gamma(v)$, we need to recall its construction.

We first note that f is étale locally trivial. Indeed we have an étale morphism $f^{-1}(0) \times \text{Pic}^0(X) \rightarrow M_H(v)$ by sending $(E, L) \in f^{-1}(0) \times \text{Pic}^0(X)$ to $E \otimes L$, and hence the composed morphism $\{E_0\} \times \text{Pic}^0(X) \rightarrow M_H(v) \rightarrow \text{Pic}^0(X)$ is the multiplication by $\text{rk } E_0$. Hence we have an identification $i_x : f^{-1}(x) \cong f^{-1}(0)$. Assume that $\text{ch}(\alpha) = (H + (H, \delta)\varrho_X)(\text{td}_X^{-1})^\vee$. Then $\mathcal{L}(n\alpha)|_{f^{-1}(x)}$ is independent of $x \in \text{Pic}^0(X)$ under the identification i_x . By the base change theorem, $f_*(\mathcal{L}(n\alpha))$ ($n \gg 0$) is locally free and $f_*(\mathcal{L}(n\alpha))_s \cong f_*(\mathcal{L}(n\alpha)_s)$. Since $\mathcal{L}(n\alpha)_s$ is base point free, we have a surjective homomorphism $f^*(f_*(\mathcal{L}(n\alpha))) \rightarrow \mathcal{L}(n\alpha)$, which implies we have a morphism $M_H^\gamma(v) \rightarrow \mathbb{P}(f_*(\mathcal{L}(n\alpha)))$ over $\text{Pic}^0(X)$ for $n \gg 0$. Then the image is the Uhlenbeck moduli space.

5.2. Wall crossing for Gieseker and Uhlenbeck moduli spaces. Assume that $H \in \text{Amp}(X)$. Since the set of walls is locally finite (Lemma 3.4), in a small neighborhood U of $H + (H, \delta)\varrho_X$, we may assume that all walls pass through $H + (H, \delta)\varrho_X$. Then $\mathcal{M}_{(\beta, tH)}(v) = \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v)$ if $\xi(\beta, tH) \in U$ and $(\beta, H) < (\delta, H)$. We classify such walls. For $H + (H, \delta)\varrho_X$, $v_1 := r_1 + \xi_1 + a_1\varrho_X \in (H + (H, \delta)\varrho_X)^\perp$ if and only if $(\xi_1/r_1, H) = (\delta, H)$. We set $v_2 := v - v_1$. Assume that $\xi(\beta, tH) \in v_1^\perp$. We may assume that $\mathcal{M}_{(\beta, tH)}(v_1) \neq \emptyset$ and $\mathcal{M}_{(\beta, tH)}(v_2) \neq \emptyset$. By shrinking a neighborhood U , we may also assume that there is no wall for v_1, v_2 between $\xi(\beta, tH)$ and $H + (H, \delta)\varrho_X$. Then $\mathcal{M}_{(\beta, tH)}(v_i) = \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v_i)$ for $i = 1, 2$. In particular, $(v_1^2), (v_2^2) \geq 0$. $u = \xi(\beta, tH) \in v_1^\perp$ means that

$$\begin{aligned} (5.10) \quad & \chi \left(e^{\beta - \frac{1}{2}K_X} - \frac{\chi(e^{\beta - \frac{1}{2}K_X}, v)}{r} \varrho_X, v_1 \right) \\ &= - (e^\beta, v_1) + \frac{r_1}{r} (e^\beta, v) = 0 \end{aligned}$$

by (1.9), (5.8) and $(H + (H, \delta)\varrho_X, v_1) = 0$. Thus

$$\frac{\chi(e^{\beta - \frac{1}{2}K_X}, v_1)}{r_1} = \frac{\chi(e^{\beta - \frac{1}{2}K_X}, v)}{r}.$$

Hence if u crosses a wall v_1^\perp , then twisted semi-stability changes.

Therefore the wall crossing of Gieseker semi-stability is naturally understood by the wall crossing of Bridgeland's stability: For a family of vectors

$$u_H := n(H + (H, \delta)\varrho_X) - \left(e^\delta + \frac{(v^2)}{2r^2} \varrho_X \right),$$

$H \in U \subset \text{Amp}(X)$, we have a family of moduli spaces, where U is a compact subset and n is a sufficiently large integer depending on U . If H crosses a wall for μ -semi-stability, then u_H crosses all walls for twisted semi-stability. If H_0 is on a wall and let H_{\pm} be ample divisors in adjacent chambers, then we have morphisms $N_{H_{\pm}}(v) \rightarrow N_{H_0}(v)$ constructed by Hu and Li [11] (see also Remark 5.5). On the other hand, for the Gieseker moduli spaces, we have a sequence of flops as was constructed by Ellingsrud-Göttsche [7], Friedman-Qin [9], Matsuki-Wentworth [17].

5.3. The case of a blow-up. Assume that $k := (c_1(v), C)$ is normalized as $0 \leq k < r$. We take $\beta_0 := \delta + p_0 C$ such that $(\beta_0, C) = 0$, that is, $\beta_0 = \pi^*(\beta')$ ($\beta' \in \text{NS}(Y)_{\mathbb{Q}}$). Let U be the open neighborhood of (β_0, H) in subsection 4.2. We take p_1 with $p_0 + \frac{1}{2} \gg p_1 > p_0$. Then we may assume that

$$(5.11) \quad \{\xi(\beta_0 + sH, H - qC) \mid \epsilon \leq 0, q > 0\} \supset \{e^{\delta}(H + xC + y(1 + \frac{v^2}{2r^2})\varrho_X)) \in \xi(U) \mid x < p_1 y, y \leq 0\},$$

where $\epsilon := s(H^2) + p_0 q$. Then $\mathcal{M}_{(\beta_0 + sH, H - qC)}(v) = \mathcal{M}_{(\beta, tH)}(v)$ for $q \geq 0$ and $\epsilon \leq 0$, where $\xi(\beta, tH) = \xi(\beta_0 + sH, H - qC)$. In particular the wall crossing in U covers the wall crossing for the moduli of perverse coherent sheaves in [22].

We first look at the wall crossing along ϱ_X^{\perp} . We note that $\sigma_{(\beta_0, H)}$ is on the wall defined by ϱ_X and also on walls defined by $v(\mathcal{O}_C(-n))$. We define $v' \in H^*(Y, \mathbb{Q})$, $\delta' \in \text{NS}(Y)_{\mathbb{Q}}$ and H' by

$$(5.12) \quad \pi^*(v') = v + kv(\mathcal{O}_C), \quad \delta' := \frac{c_1(v')}{r}, \quad H = \pi^*(H').$$

Lemma 5.7. (1) $\mathcal{M}_{(\beta_0, H)}(v)$ consists of E which is S -equivalence to $\pi^*(F) \oplus A$ where F is a locally free poly-stable sheaf F on Y and

$$(5.13) \quad A = \bigoplus_{i=1}^m k_{x_i}[-1] \oplus \mathcal{O}_C[-1]^{\oplus n_1} \oplus \mathcal{O}_C(-1)^{\oplus n_2}, \quad (x_i \in X \setminus C).$$

(2) The S -equivalence class of E in (1) is uniquely determined by the S -equivalence class of $\mathbf{R}\pi_*(E) \in \mathcal{M}_{(\delta', H')}(v')$, that is, it is determined by F and

$$(5.14) \quad A' := \mathbf{R}\pi_*(A) = \bigoplus_{i=1}^m k_{x_i}[-1] \oplus k_p[-1]^{\oplus n_1}, \quad (x_i \in X \setminus C).$$

Proof. (2) By A' , $\{x_1, \dots, x_m\}$ and n_1 are determined. Since $c_1(E) = c_1(\pi^*(F)) + (n_2 - n_1)C$, n_2 is also determined. Hence the S -equivalence class of E is uniquely determined by the S -equivalence class of $\mathbf{R}\pi_*(E)$. \square

For $\sigma_{(\beta_0 + sH, H - qC)}$ with $s(H^2) + p_0 q = 0$ and $\sigma_{(\beta_0, H)}$ on ϱ_X^{\perp} , we have a contraction $N_{H - qC}(v) \rightarrow N_{H'}(v')$.

Lemma 5.8. For $E \in \mathfrak{C}^0$ with $\text{Hom}(\mathcal{O}_C, E) = 0$, there is an exact sequence

$$(5.15) \quad 0 \rightarrow E \rightarrow \mathbf{L}\pi^*(F) \rightarrow \text{Ext}^1(\mathcal{O}_C, E) \otimes \mathcal{O}_C \rightarrow 0.$$

Proof. We note that $\text{Ext}^2(\mathcal{O}_C, E) = \text{Hom}(E, \mathcal{O}_C(-1))^{\vee} = 0$. Hence

$$\dim \text{Ext}^1(\mathcal{O}_C, E) = -\chi(\mathcal{O}_C, E) = (c_1(E), C).$$

Then the universal extension E' is an object of \mathfrak{C}^0 such that $\text{Hom}(\mathcal{O}_C, E') = 0$ and $(c_1(E'), C) = 0$. Since $F := \pi_*(E')$ is torsion free, $\mathbf{L}\pi^*(F) = \pi^*(F)$ and $\phi : \pi^*(F) \rightarrow E'$ is surjective in \mathfrak{C}^0 . By comparing the first Chern classes, we see that ϕ is isomorphic. \square

Remark 5.9. Since $\mathbf{R}\pi_*(\mathcal{O}_X(C)) = \mathcal{O}_Y$, $F = \mathbf{R}\pi_*(\mathbf{L}\pi^*(F)(C)) = \pi_*(E(C))$.

In $\mathcal{A}_{(\beta' + sH', H')}(X/Y)$, (5.15) gives an exact sequence

$$(5.16) \quad 0 \rightarrow \text{Hom}(\mathcal{O}_C[-1], E) \otimes \mathcal{O}_C \rightarrow E \rightarrow \mathbf{L}\pi^*(F) \rightarrow 0.$$

Therefore we get the following.

Proposition 5.10. (1) If $k \neq 0$, then $\mathcal{M}_{(\beta_0, H - qC)}(v) = \emptyset$ for $q < 0$.

(2) If $k = 0$, then $\mathcal{M}_{(\beta_0, H - qC)}(v) = \mathcal{M}_{(\delta', H')}(v')$ for $q < 0$.

We next consider the wall crossing for the Gieseker semi-stability. For simplicity, we assume that $\gcd(r, (c_1(v), H)) = 1$. In this case, the hyperplanes $v(\mathcal{O}_C(-n))^{\perp}$ ($n > 0$) in $C^+(v)$ are the candidates of walls. Let \mathcal{C}_n be a chamber between $v(\mathcal{O}_C(-n))^{\perp}$ and $v(\mathcal{O}_C(-n-1))^{\perp}$. Then $n - \frac{1}{2} < -(\beta, C) < n + \frac{1}{2}$ for $\xi(\beta, \omega) \in \mathcal{C}_n$ with $\omega \in \mathbb{R}_{>0}H$ and $\mathcal{M}_{\mathcal{C}_n}(v) \cong \mathcal{M}_H^{\beta - \frac{1}{2}K_X}(v)$. By Lemma 2.5 or the finiteness of walls in U , there is an integer $N(v)$ such that $\mathcal{M}_{\mathcal{C}_n}(v) = \mathcal{M}_{H - qC}^{\beta - \frac{1}{2}K_X}(v)$ for $n \geq N(v)$ where $q > 0$ is sufficiently small. If $k = (c_1(v), C) = 0$, then $\mathcal{M}_{\mathcal{C}_0}(v)$ is the moduli space of semi-stable sheaves on Y . Thus the wall-crossing in [22] is the wall crossing in the space of stability conditions. If $0 < k < r$, then by (5.16), we have a morphism

$$\mathcal{M}_{\mathcal{C}_0}(v) \rightarrow \mathcal{M}_{(\beta' + sH', H')}(v')$$

whose general fibers are $Gr(r, k)$ -bundles. If $\xi(\beta_0 + sH, H - qC) \in \mathcal{C}_0$ crosses the wall $v(\mathcal{O}_C)^\perp$, then $\mathcal{M}_{(\beta_0 + sH, H - qC)}(v) = \emptyset$ for $q < 0$.

The relation between $M_{\mathcal{C}_0}(v)$ and $M_{\mathcal{C}_{N(v)}}(v)$ is described as Mumford-Thaddeus type flips:

$$(5.17) \quad \begin{array}{ccccccc} M_{\mathcal{C}_0}(v) & & M_{\mathcal{C}_1}(v) & & M_{\mathcal{C}_{N(v)}}(v) \\ & \searrow & \swarrow & \searrow & \swarrow \\ & M_{\mathcal{C}_{0,1}}(v) & & M_{\mathcal{C}_{1,2}}(v) & \end{array} \quad \dots$$

where $M_{\mathcal{C}_{n,n+1}}(v)$ parameterizes S -equivalence classes of $\sigma_{(\beta,\omega)}$ -twisted semi-stable objects with $\xi(\beta, \omega) \in v(\mathcal{O}_C(-n-1))^\perp$.

Remark 5.11. Although we considered the wall-crossing behavior in a neighborhood of (β_0, H) , we may move the parameter p as in [22]. In \mathcal{C}_n , $n \geq 0$, the wall-crossing behaviour is the same. However for the wall $v(\mathcal{O}_C)^\perp$, the behavior is different. For $E \in \mathcal{M}_{\mathcal{C}_0}(v)$, we have an exact sequence

$$(5.18) \quad 0 \rightarrow E' \rightarrow E \rightarrow \text{Hom}(E, \mathcal{O}_C)^\vee \otimes \mathcal{O}_C \rightarrow 0$$

with $E' \in \mathfrak{C}^1 \cap \mathfrak{C}^0$. If $k > 0$, then we have a morphism $M_{\mathcal{C}_0}(v) \rightarrow M_{(\beta', H')}(w')$ by sending E to $\pi_*(E(C))$, where $\pi^*(w') = v - (r - k)v(\mathcal{O}_C)$.

6. APPENDIX

6.1. Some properties of perverse coherent sheaves. Let $\pi : X \rightarrow Y$ be a resolution of a rational singularity. Let \mathfrak{C} be a category of perverse coherent sheaves and G be a local projective generator of \mathfrak{C} which is a locally free sheaf on X ([28, Defn. 1.1.3]).

Lemma 6.1. *For a torsion free object E of \mathfrak{C} , there is an exact sequence*

$$(6.1) \quad 0 \rightarrow E \rightarrow E' \rightarrow T \rightarrow 0$$

such that T is 0-dimensional, E' is torsion free and $\text{Ext}^1(A, E') = 0$ for all 0-dimensional objects A of \mathfrak{C} .

Proof. If $\text{Ext}^1(A, E) \neq 0$ for an irreducible object A of \mathfrak{C} , then a non-trivial extension is torsion free. So we inductively construct torsion free objects E_n

$$0 \subset E = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_n$$

such that E_i/E_{i-1} are irreducible objects of \mathfrak{C}^β . We set $T_n := E_n/E_0$. We have an exact sequence

$$(6.2) \quad 0 \rightarrow \pi_*(G^\vee \otimes E) \rightarrow \pi_*(G^\vee \otimes E_n) \rightarrow \mathbf{R}\pi_*(G^\vee \otimes T_n) \rightarrow 0.$$

By the torsion freeness of E, E_n , $\pi_*(G^\vee \otimes E)$ and $\pi_*(G^\vee \otimes E_n)$ are torsion free sheaves on Y by

$$(6.3) \quad \text{Hom}(\mathbb{C}_y, \pi_*(G^\vee \otimes E_n)) = \text{Hom}(G \otimes \mathcal{O}_{\pi^{-1}(y)}, E_n) = 0, \quad y \in Y.$$

Hence $\pi_*(G^\vee \otimes E_n)$ is regarded as a subsheaf of $\pi_*(G^\vee \otimes E)^{\vee\vee}$ and

$$\chi(\pi_*(G^\vee \otimes E)^{\vee\vee} / \pi_*(G^\vee \otimes E)) \geq \chi(\mathbf{R}\pi_*(G^\vee \otimes T_n)) = \chi(G, T_n).$$

On the other hand, $\chi(G, T_i/T_{i-1}) > 0$ for all i imply that $\chi(G, T_n) \geq n$. Therefore n is bounded above. Hence there is n such that E_n satisfies the desired property. \square

Lemma 6.2. *For a μ -semi-stable object E of \mathfrak{C} , $d_\beta(E)^2(H^2) - 2 \text{rk } E a_\beta(E) \geq 0$.*

Proof. We take E' in Lemma 6.1. Then E' is a μ -semi-stable locally free sheaf with respect to H , since $\text{Ext}^1(k_x, E') = 0$ for all $x \in X$ (cf. [28, Lem. 1.1.31]). Since $a_\beta(T) \geq 0$, we have

$$d_\beta(E)^2(H^2) - 2 \text{rk } E a_\beta(E) \geq d_\beta(E')^2(H^2) - 2 \text{rk } E' a_\beta(E').$$

Then the claim follows from the ordinary Bogomolov-Gieseker inequality. \square

The following weak form of Bogomolov inequality is an easy consequence of Lemma 6.2 and the proof of Lemma 2.20.

Proposition 6.3. *For a $\sigma_{(\beta,\omega)}$ -semi-stable object E of $\mathcal{A}_{(\beta,\omega)}$,*

$$(6.4) \quad d_\beta(E)^2(H^2) - 2 \text{rk } E a_\beta(E) \geq 0.$$

We shall postpone the proof of this claim, since we need a claim in Proposition 6.11. So we shall prove (6.4) in the course of the proof of Proposition 6.11.

Let $\mathfrak{C}^* = \mathfrak{C}^D(-K_X)$ be a category of perverse coherent sheaves such that $G^\vee(-K_X)$ is a local projective generator (see [28, Lem. 1.1.14]). Since $(G^\vee(-K_X))^\vee(-K_X) = G$, we have $(\mathfrak{C}^*)^* = \mathfrak{C}$. By [28, Lem. 1.1.14 (4)], the following claim holds.

Lemma 6.4. *$A \in \mathbf{D}(X)$ is a 0-dimensional object of \mathfrak{C} if and only if $A^\vee[2]$ is a 0-dimensional object of \mathfrak{C}^* .*

For a 0-dimensional object A of \mathfrak{C} ,

$$(6.5) \quad \begin{aligned} \mathrm{Hom}(A, E) &= \mathrm{Hom}(E^\vee[1], (A^\vee[2])[-1]), \\ \mathrm{Hom}(E, A[-1]) &= \mathrm{Hom}(A^\vee[2], E^\vee[1]). \end{aligned}$$

Lemma 6.5. *Let $E \in \mathbf{D}(X)$ satisfy ${}^p H^i(E) = 0$ ($i \neq -1, 0$) and $\mathrm{Hom}(\mathbb{C}_x, E) = 0$ for all $x \in X$. Then $H^i(E^\vee[1]) = 0$ for $i \neq -1, 0$ and $\mathrm{Hom}(B, E^\vee[1]) = 0$ for all 0-dimensional object B of \mathfrak{C}^* . Moreover if $\mathrm{Hom}(A, E) = 0$ for all 0-dimensional object A of \mathfrak{C} , then ${}^p H^i(E^\vee[1]) = 0$ for $i \neq -1, 0$.*

Proof. Let G be a local projective generator of \mathfrak{C} . Since $\mathrm{Ext}^2(G(-n), {}^p H^{-1}(E)) = 0$ for $n \gg 0$, we have a morphism $G(-n)^{\oplus N} \rightarrow E$ such that ${}^p H^0(G(-n)^{\oplus N}) \rightarrow {}^p H^0(E)$ is surjective in \mathfrak{C} . We set $V_0 := G(-n)^{\oplus N}$ and $V_{-1} := \mathrm{Cone}(V_0 \rightarrow E)[-1]$. Then we see that $V_{-1} \in \mathfrak{C}$. Since V_0 is a locally free sheaf, $\mathrm{Ext}^1(\mathbb{C}_x, V_0) = 0$ for all $x \in X$. By $\mathrm{Ext}^2(E, \mathbb{C}_x) = \mathrm{Hom}(\mathbb{C}_x, E)^\vee = 0$, $\mathrm{Ext}^1(V_{-1}, \mathbb{C}_x) = 0$ for all $x \in X$. Hence V_{-1} is a locally free sheaf. Then $E^\vee[1]$ is represented by a two term complex of locally free sheaves. For a 0-dimensional object B of \mathfrak{C}^* , we set $A := B^\vee[2]$. Then $A^\vee[2] = B$ and Lemma 6.4 implies that A is a 0-dimensional object of \mathfrak{C} . Then (6.5) implies $\mathrm{Hom}(B, E^\vee[1]) = \mathrm{Hom}(E, A[-1]) = 0$.

Assume that $\mathrm{Hom}(A, E) = 0$ for all 0-dimensional objects. Then by Lemma 6.4 and (6.5), we see that $H^0(E^\vee[1]) \in \mathfrak{C}^*$. For $F := H^{-1}(E^\vee[1])$, we have an exact triangle

$$(6.6) \quad {}^p H^0(F) \rightarrow F \rightarrow {}^p H^1(F)[-1] \rightarrow {}^p H^0(F)[1].$$

Therefore ${}^p H^i(E^\vee[1]) = 0$ for $i \neq -1, 0$, ${}^p H^{-1}(E^\vee[1]) = {}^p H^0(F)$ and we have an exact sequence

$$0 \rightarrow {}^p H^1(F) \rightarrow {}^p H^0(E^\vee[1]) \rightarrow H^0(E^\vee[1]) \rightarrow 0$$

in \mathfrak{C}^* . □

Remark 6.6. V_i^\vee are local projective objects of \mathfrak{C}^D . However we do not know whether $V_i^\vee \in \mathfrak{C}^*$ or not.

Corollary 6.7. *For a complex $E \in \mathbf{D}(X)$, ${}^p H^i(E) = 0$ ($i \neq -1, 0$) and $\mathrm{Hom}(A, E) = 0$ for all 0-dimensional object A of \mathfrak{C} if and only if ${}^p H^i(E^\vee[1]) = 0$ for $i \neq -1, 0$ and $\mathrm{Hom}(B, E^\vee[1]) = 0$ for all 0-dimensional object B of \mathfrak{C}^* .*

- Remark 6.8.* (1) Let E be a σ -stable object with $0 < \phi(E) < 1$. Then $\mathrm{Hom}(A, E) = 0$ for all 0-dimensional object A of \mathfrak{C} .
(2) For the object E in Lemma 6.5, if $\mathrm{rk} {}^p H^0(E) = 0$, then $E^\vee[1] \in \mathrm{Coh}(X)$. Moreover if $\mathrm{Hom}(A, E) = 0$ for all 0-dimensional object A of \mathfrak{C} , then $E^\vee[1] \in \mathfrak{C}^*$ and $E^\vee[1]$ does not contain any 0-dimensional object.
(3) Let $V_{-1} \rightarrow V_0$ be a two term complex of locally free sheaves. Then ${}^p H^i(E) = 0$ for $i \neq 0, -1$ if $\mathrm{Hom}(E, A[-1]) = 0$ for all 0-dimensional object A of \mathfrak{C} .

Lemma 6.9. *Let $\pi : X \rightarrow Y$ be the blow-up of a point of a smooth surface Y and C the exceptional divisor. Assume that $l - \frac{1}{2} < (\beta, C) < l + \frac{1}{2}$.*

- (1) $\mathcal{O}_C(l)$ is an irreducible object of $\mathcal{A}_{(\beta, tH)}$.
(2) Let E be a subobject of $\mathcal{O}_C(l)^{\oplus n}$ in $\mathcal{A}_{(\beta, tH)}$. Then $E = \mathcal{O}_C(l)^m$ with $0 \leq m \leq n$.

Proof. It is sufficient to prove (2). We set $F := \mathcal{O}_C(l)^{\oplus n}/E \in \mathcal{A}_{(\beta, tH)}$. Then we have an exact sequence

$$(6.7) \quad 0 \rightarrow {}^p H^{-1}(F) \rightarrow {}^p H^0(E) \rightarrow \mathcal{O}_C(l)^{\oplus n} \rightarrow {}^p H^0(F) \rightarrow 0$$

in \mathfrak{C}^l and ${}^p H^{-1}(E) = 0$. Since $(c_1({}^p H^{-1}(F)(-\beta)), H) \leq 0$ and $(c_1({}^p H^0(E)(-\beta)), H) \geq 0$, we see that ${}^p H^0(E)$ is 0-dimensional and ${}^p H^{-1}(F) = 0$. Since $\mathcal{O}_C(l)$ is an irreducible object of \mathfrak{C}^l , we see that $E = \mathcal{O}_C(l)^{\oplus m}$. □

Corollary 6.10. *For $E \in \mathfrak{C}^\beta$, there is a filtration*

$$0 \subset F_1 \subset F_2 \subset F_3 = E$$

such that $F_1 = \mathcal{O}_C(l)^{\oplus n_1}$, $\mathrm{Hom}(\mathcal{O}_C(l), F_2/F_1) = \mathrm{Hom}(F_2/F_1, \mathcal{O}_C(l)) = 0$ and $F_3/F_2 = \mathcal{O}_C(l)^{\oplus n_3}$.

6.2. Large volume limit. We consider the stability condition $\sigma_{(\beta, \omega)}$ in 1.3. This stability condition is very similar to that for an abelian surface. Thus almost same results in [18],[19] and [25] hold. In particular results in [25, sect. 4.1] hold.

We set

$$v := e^\beta(r + a_\beta \varrho_X + d_\beta H + D_\beta), D_\beta \in H^\perp.$$

By Lemma 1.8, [18, Lem. 3.1.8] holds, where

$$(6.8) \quad (\omega^2) > \begin{cases} \frac{d_\beta - d_{\min, \beta}}{r d_0}((v^2) - (D_\beta^2)), & r > 0 \\ \frac{d_\beta - d_{\min, \beta}}{d_0}((v^2) - (D_\beta^2)) + \frac{2d_\beta |a_\beta|}{d_0}, & r = 0, \\ \frac{-d_\beta + d_{\min, -\beta}}{r d_0}((v^2) - (D_\beta^2)), & r < 0, \end{cases}$$

$d_{\min, \beta} := \min\{d_\beta(E) > 0 \mid E \in K(X)\}$ and $d_0 = \frac{1}{(H^2)} \min\{(D, H) > 0 \mid D \in \text{Pic}(X)\}$.

We set $\beta = \xi/r'$, $\xi \in \text{NS}(X)$. Then

$$(6.9) \quad (c_1(E) - r\beta, H) = \frac{(r'c_1(E) - r\xi, H)}{r'} \in \mathbb{Z} \frac{d_0(H^2)}{r'}.$$

If $\gcd(\frac{(c_1(E), H)}{d_0(H^2)}, r) = 1$, that is, $r' \frac{(c_1(E), H)}{d_0(H^2)} - ra = 1$ ($r', a \in \mathbb{Z}$), then for $\xi \in \text{NS}(X)$ with $(\xi, H) = ad_0(H^2)$, we have

$$(c_1(E) - r\xi/r', H) = \frac{(r'c_1(E) - r\xi, H)}{r'} = \frac{d_0(H^2)}{r'} = d_{\min, \xi/r'}(H^2).$$

By the same proof of [18, Prop. 3.2.1, Prop. 3.2.7], we have the following results.

Proposition 6.11. *Assume that ω satisfies (6.8) with respect to v . Then $E \in \mathbf{D}(X)$ is $\sigma_{(\beta, \omega)}$ -semi-stable with $\phi(E) \in (0, 1)$ if and only if*

- (i) $\text{rk } v \geq 0$, $E \in \mathfrak{C}$ and is $(\beta - \frac{1}{2}K_X)$ -twisted semi-stable or
- (ii) $\text{rk } v < 0$, $E^\vee[1] \in \mathfrak{C}^*$ and is $-(\beta + \frac{1}{2}K_X)$ -twisted semi-stable.

Thus we have

$$(6.10) \quad \mathcal{M}_{(\beta, \omega)}(v) \cong \begin{cases} \mathcal{M}_\omega^{\beta - \frac{1}{2}K_X}(v), & \text{rk } v \geq 0, \\ \mathcal{M}_\omega^{-\beta - \frac{1}{2}K_X}(-v^\vee), & \text{rk } v < 0. \end{cases}$$

Proof. Let E be a $\sigma_{(\beta, \omega)}$ -semi-stable object for $(\omega^2) \gg 0$. By using Lemma 6.2 for $(\beta - \frac{1}{2}K_X)$ -semi-stable objects of \mathfrak{C} and $-(\beta + \frac{1}{2}K_X)$ -semi-stable object of \mathfrak{C}^* , we see that E is $(\beta - \frac{1}{2}K_X)$ -semi-stable or $E^\vee[1]$ is $-(\beta + \frac{1}{2}K_X)$ -semi-stable ([18, Prop. 3.2.1, Lem. 3.2.3, Prop. 3.2.7]). In particular, E satisfies Proposition 6.3. For a general ω , we use

$$(6.11) \quad \frac{((v_1 + v_2)^2) - (D_1 + D_2)^2}{(d_1 + d_2)^2} = \frac{(v_1^2) - (D_1^2)}{d_1^2} + \frac{(v_2^2) - (D_2^2)}{d_2^2} + \left(\frac{a_1}{d_1} - \frac{a_2}{d_2}\right) \left(\frac{r_1}{d_1} - \frac{r_2}{d_2}\right)$$

for

$$v_i := e^\beta(r_i + d_i H + D_i + a_i \varrho_X), \quad D_i \in H^\perp.$$

to study the wall crossing behavior. Then Proposition 6.3 follows by the induction on d_β .

Thanks to Proposition 6.3, the converse direction also holds. For more detail, see [18]. \square

Remark 6.12. If $\mathfrak{C}^\beta = \text{Coh}(X)$, then the claim is a refinement of results of Lo and Qin [15] or Maciocia [16].

The following claim is due to Maciocia [16, Thm. 3.13] if $\mathfrak{C}^\beta = \text{Coh}(X)$

Lemma 6.13 ([18, sect. 4]). *We fix $\beta \in \text{NS}(X)_\mathbb{Q}$. Assume that $d_\beta(v) > 0$ and*

$$(6.12) \quad v_1 \in \mathbb{Q}v \iff (r(v_1), d_\beta(v_1), a_\beta(v_1)) \in \mathbb{Q}(r(v), d_\beta(v), a_\beta(v))$$

for all $v_1 \in H^*(X, \mathbb{Q})_{\text{alg}}$ with $0 < d_\beta(v_1) < d_\beta(v)$. Then the set of walls intersecting $\{(\beta, tH) \mid t > 0\}$ is finite.

Proof. Let $B \in \mathbb{Z}_{>0}$ be the denominator of β . Assume that we have a decomposition $v = v_1 + v_2$ such that there are $\sigma_{(\beta, \omega)}$ -semi-stable objects E_i with $v(E_i) = v_i$ ($i = 1, 2$) and $Z_{(\beta, \omega)}(v_1), Z_{(\beta, \omega)}(v_2) \in \mathbb{R}_{>0}Z_{(\beta, \omega)}(v)$. We set

$$(6.13) \quad \begin{aligned} v &:= e^\beta(r + dH + D + a\varrho_X), \quad D \in H^\perp \\ v_i &:= e^\beta(r_i + d_i H + D_i + a_i \varrho_X), \quad D_i \in H^\perp. \end{aligned}$$

Then

$$(6.14) \quad \frac{(\omega^2)}{2}(dr_i - d_i r) = (da_i - d_i a).$$

By Lemma 6.2, $(v_i^2) - (D_i^2) = d_i^2(H^2) - 2r_i a_i \geq 0$. Since

$$(6.15) \quad \frac{(v_1, v_2) - (D_1, D_2)}{d_1 d_2} = \frac{(v_1^2) - (D_1^2)}{2d_1^2} + \frac{(v_2^2) - (D_2^2)}{2d_2^2} + \left(\frac{r_1}{d_1} - \frac{r_2}{d_2}\right) \left(\frac{a_1}{d_1} - \frac{a_2}{d_2}\right),$$

$(v_1, v_2) - (D_1, D_2) > 0$. Hence $0 \leq (v_i^2) - (D_i^2) < (v^2) - (D^2)$ and $(v_i, v - v_i) - (D_i, D - D_i) > 0$. Since the choice of d_i are finite, $r_i a_i$ are bounded. $(v^2) - (D^2) \geq 2((v_i, v) - (D_i, D)) > 0$ implies $r a_i + r_i a$ is also bounded. Therefore r_i, a_i are bounded. Since $r_i \in \mathbb{Z}$ and $2B^2 a_i \in \mathbb{Z}$, the choice of r_i and a_i are finite. Therefore the set of walls is finite. \square

6.3. Examples of Gieseker chamber. Let X be a surface with $\text{Pic}(X) = \mathbb{Z}H$. Let $v = (r, H, a)$, $r \geq 0$. We note that

$$(6.16) \quad \deg E = \min\{\deg E' > 0 \mid E' \in K(X)\}$$

for E with $v(E) = v$.

Lemma 6.14. *Assume that*

$$(6.17) \quad a \leq -\frac{1}{2}(K_X, H) - r - 1.$$

(1) *For all $L \in M_H(0, H, a)$, $H^0(X, L) = 0$ and $\dim \text{Ext}^1(L, \mathcal{O}_X) \geq r + 1$.*

(2) *For $L \in M_H(0, H, a)$, we set $n := \dim \text{Ext}^1(L, \mathcal{O}_X)$. Then there is a family of stable sheaves*

$$(6.18) \quad 0 \rightarrow \mathcal{O}_X \otimes V \rightarrow E \rightarrow L \rightarrow 0$$

parameterized by $\text{Gr}(n, r)$, where V is the universal subspace of dimension r .

Proof. We note that $(K_X, H) \geq (H^2)$ by $\text{NS}(X) = \mathbb{Z}H$. (1) Since $\deg L = a + \frac{(H^2)}{2} < 0$, $H^0(X, L) = 0$. Since $\text{Hom}(L, \mathcal{O}_X) = 0$ and $\chi(L, \mathcal{O}_X) = \chi(L(K_X)) = a + \frac{(H^2)}{2} + (K_X, H) + \chi(\mathcal{O}_H) = a + \frac{(K_X, H)}{2}$, we get (1).

(2) is well-known by (6.16). \square

We note that

$$\text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)_{\mathbb{R}} = \{(sH, tH) \mid s \in \mathbb{R}, t > 0\}.$$

By [16], walls are defined by circles containing $(\frac{1}{r} - \sqrt{\frac{(v^2)}{r^2(H^2)}}, 0)$ for $r > 0$ and containing $(\frac{a}{(H^2)}, 0)$. Since $(v^2) > (H^2)$, the center is in $s < 0$. By (6.16), all walls are in $s \leq 0$. For a Mukai vector $w = (r', d'H, a')$, we set

$$W_w := \{(s, t) \mid \mathbb{R}Z_{(sH, tH)}(v) = \mathbb{R}Z_{(sH, tH)}(w)\}.$$

It is a candidate of walls for v . By [25, Lem. 5.5, Prop. 5.6], $\{W_w\}$ forms a pencil of circles. Hence there is a unique W_w passing $(0, 0)$. By [25, Lem. 5.11], it is defined by $w = v(\mathcal{O}_X) = (1, 0, 0)$, and the equation is

$$t^2 + s \left(s - \frac{2}{(H^2)}a \right) = 0.$$

In particular, it is independent of the choice of r . If w defines a wall, then W_w is contained in $W_{(1,0,0)}$. In particular $\mathcal{M}_{(sH, tH)}(r, H, a) = \mathcal{M}_H(r, H, a)$ in the exterior of $W_{(1,0,0)}$. Hence all $E \in \mathcal{M}_H(r-1, H, a)$ are $\sigma_{(sH, tH)}$ -semi-stable on $W_{(1,0,0)}$. Therefore $W_{(1,0,0)}$ defines a wall for v and gives a boundary of the Gieseker chamber if a satisfies (6.17).

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